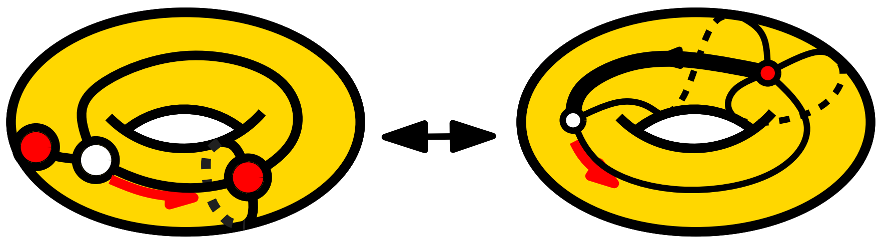
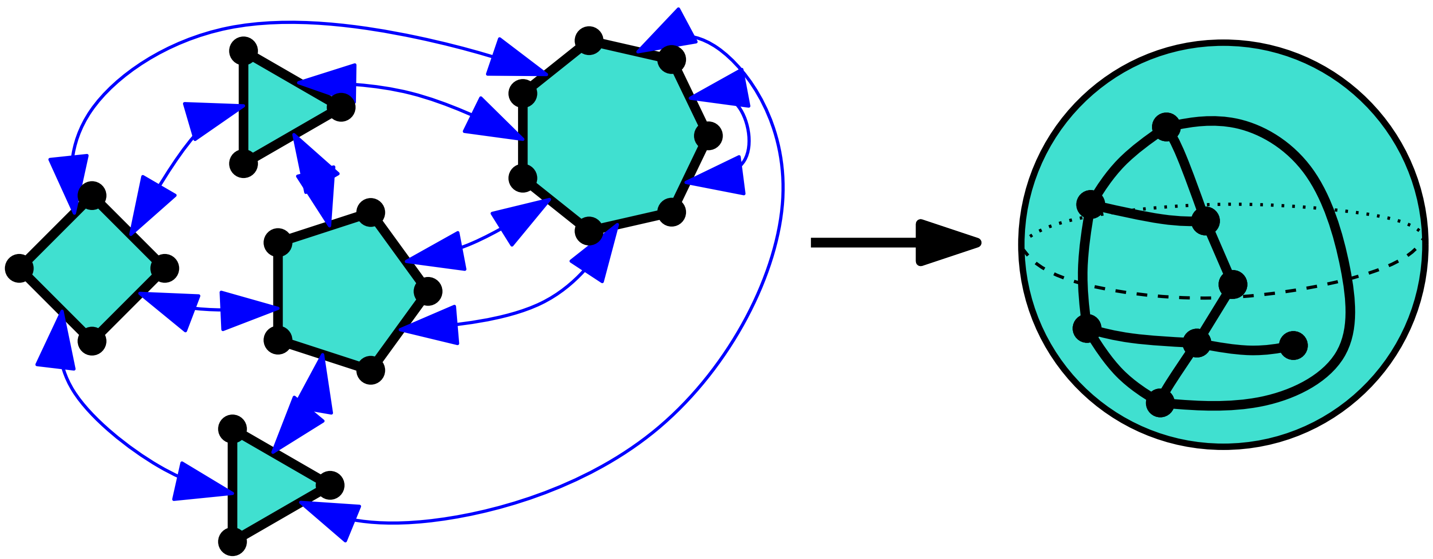
Counting one-face maps and one-face constellations

Olivier Bernardi (Brandeis University)



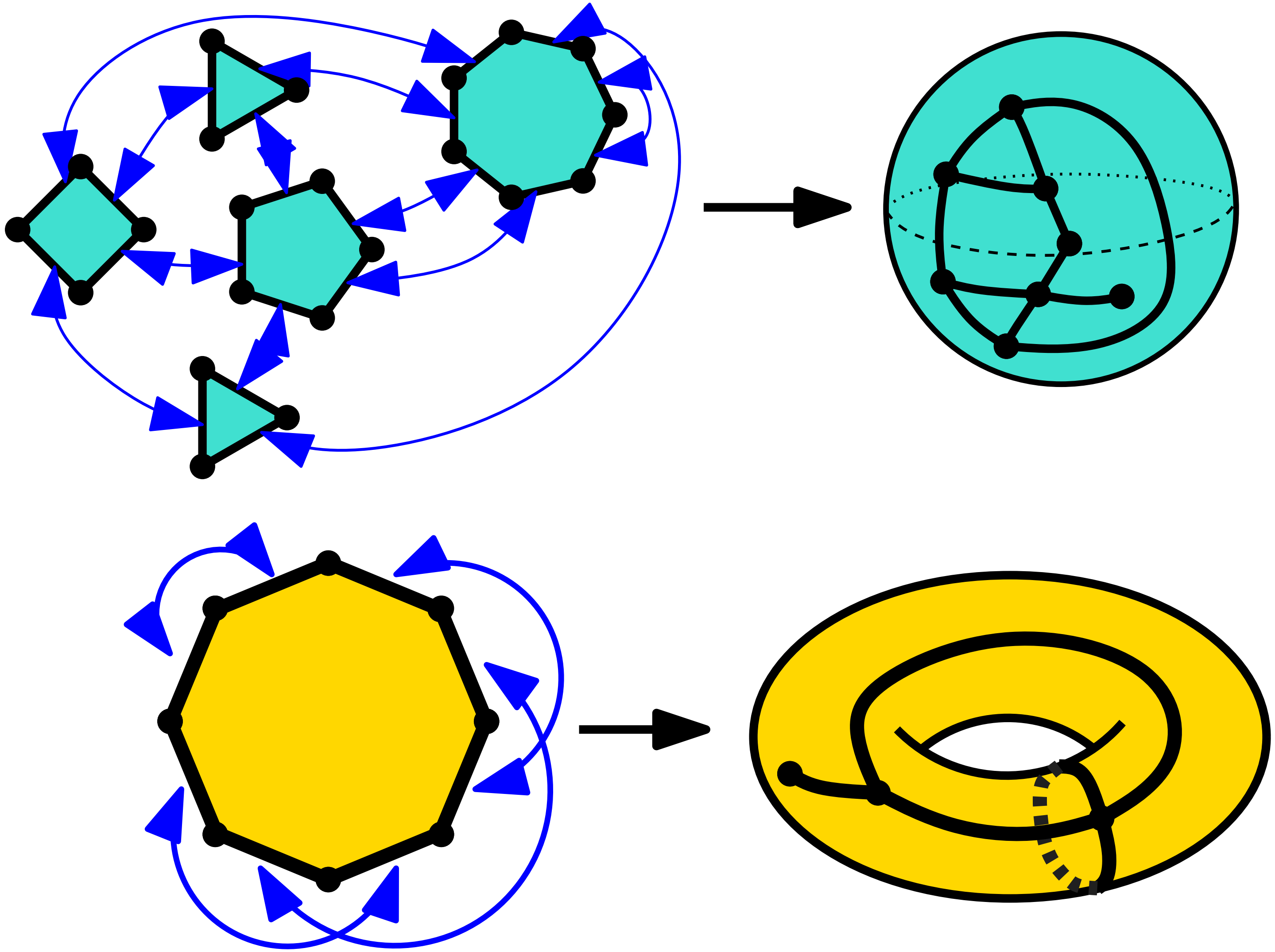
Journees cartes, June 2013

Maps



Definition

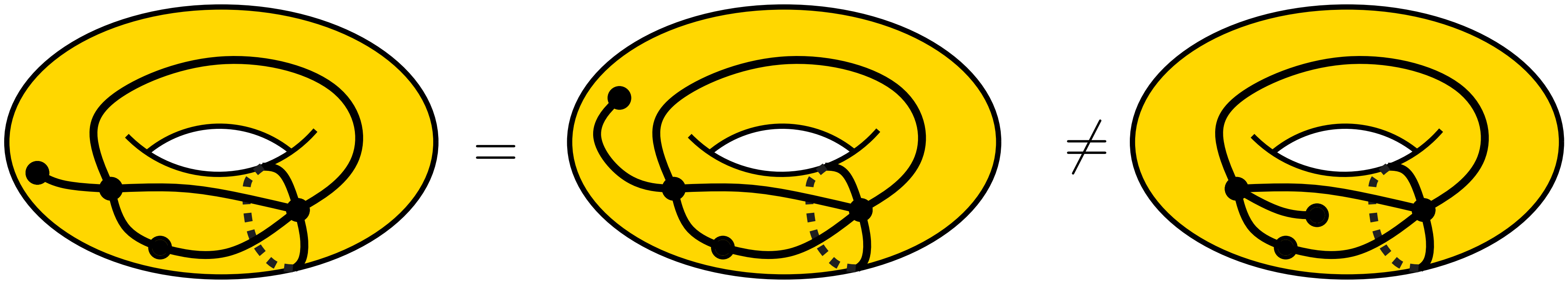
Def 1. A **map** is a gluing of polygons giving a connected surface without boundary.



Definition

Def 1. A **map** is a gluing of polygons giving a connected surface without boundary.

Def 2. A **map** is a connected graph embedded in a surface (with simply connected faces) considered up to homeomorphism.

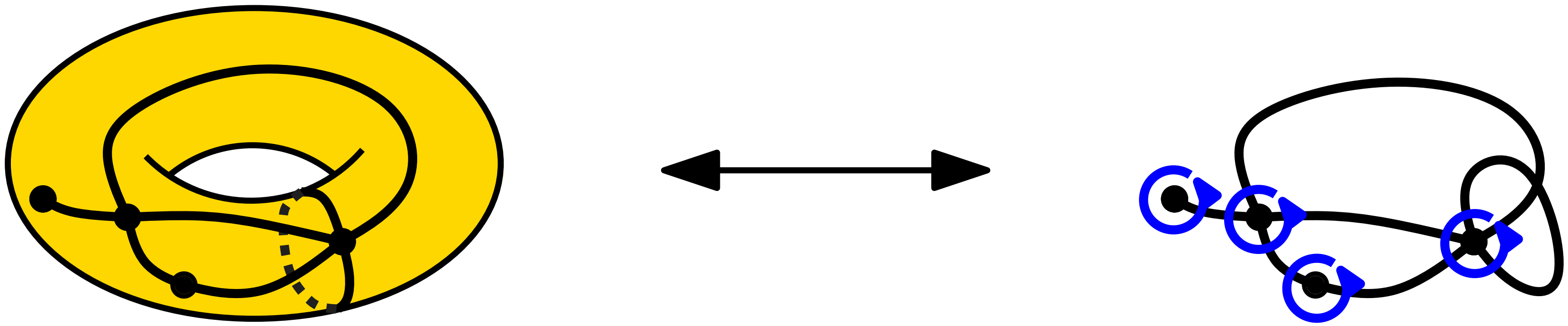


Definition

Def 1. An **orientable map** is a gluing of polygons giving a connected orientable surface without boundary.

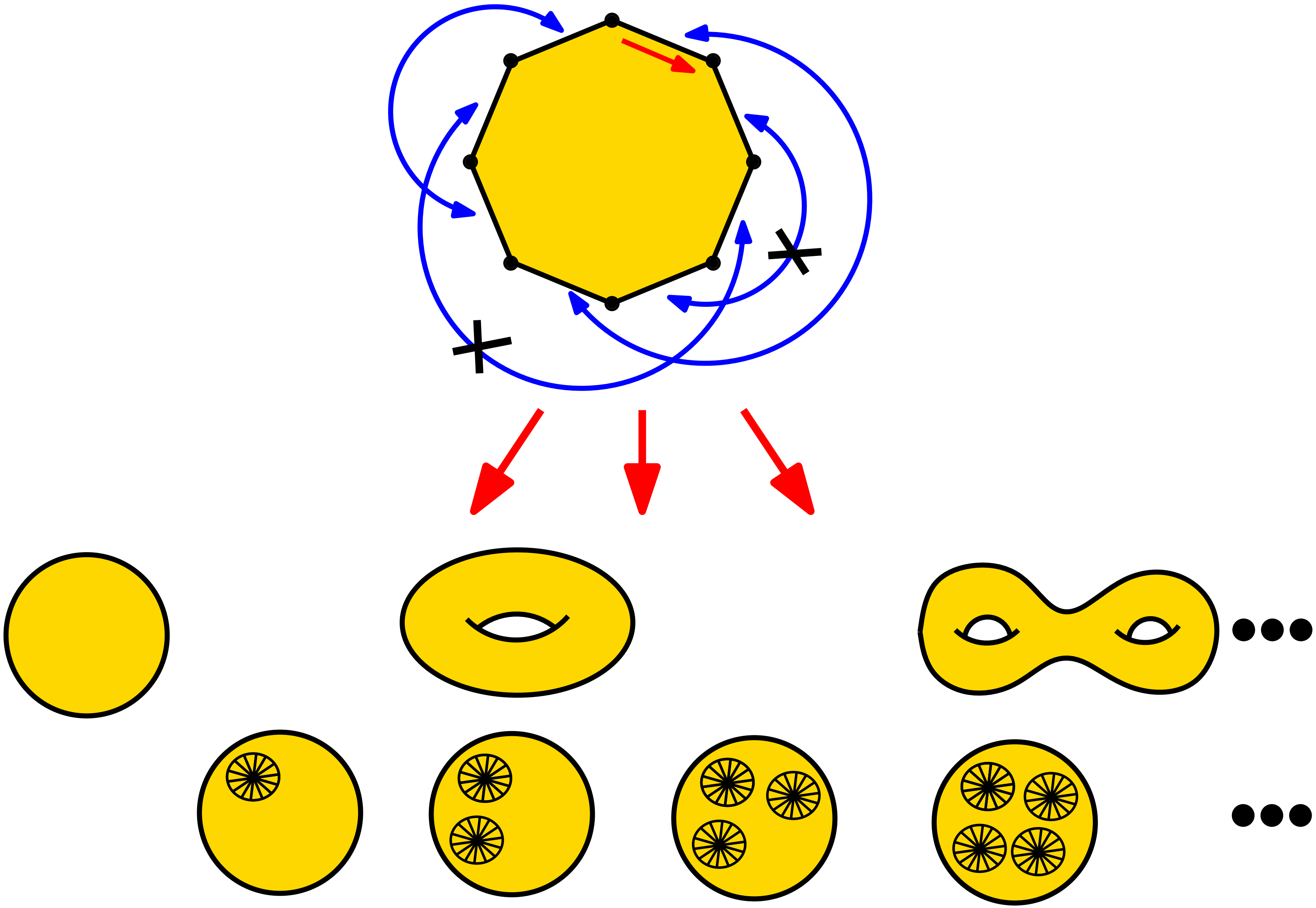
Def 2. An **orientable map** is a connected graph embedded in an orientable surface considered up to homeomorphism.

Def 3. An **orientable map** is a connected graph + a cyclic ordering of the half-edges around each vertex (the clockwise ordering).



Counting problem

Question: Among all the one-face maps obtained from a $2n$-gon, how many times do we get each surface?



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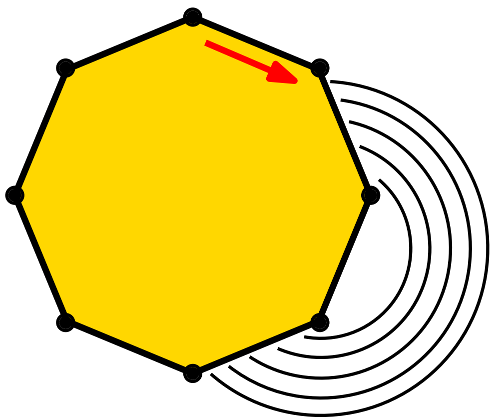
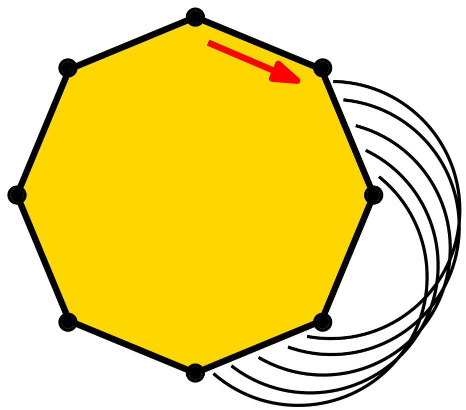
Question: Among all the one-face maps obtained from a $2n$-gon, how many times do we get each surface?

Each pair of edges can be glued in a orientable or non-orientable way. The surface is orientable if and only if each gluing is orientable.



Orientable gluing



Non-orientable gluing

$(2n-1)!! = (2n-1)(2n-3)\cdots 1$ ways of getting orientable surface.
$2^n(2n-1)!!$ ways of getting general surface.

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Question: Among all the one-face maps obtained from a $2n$-gon, how many times do we get each surface?

Remark 1. The surface obtained is characterized by its orientability and the number of vertices of the one-face map.

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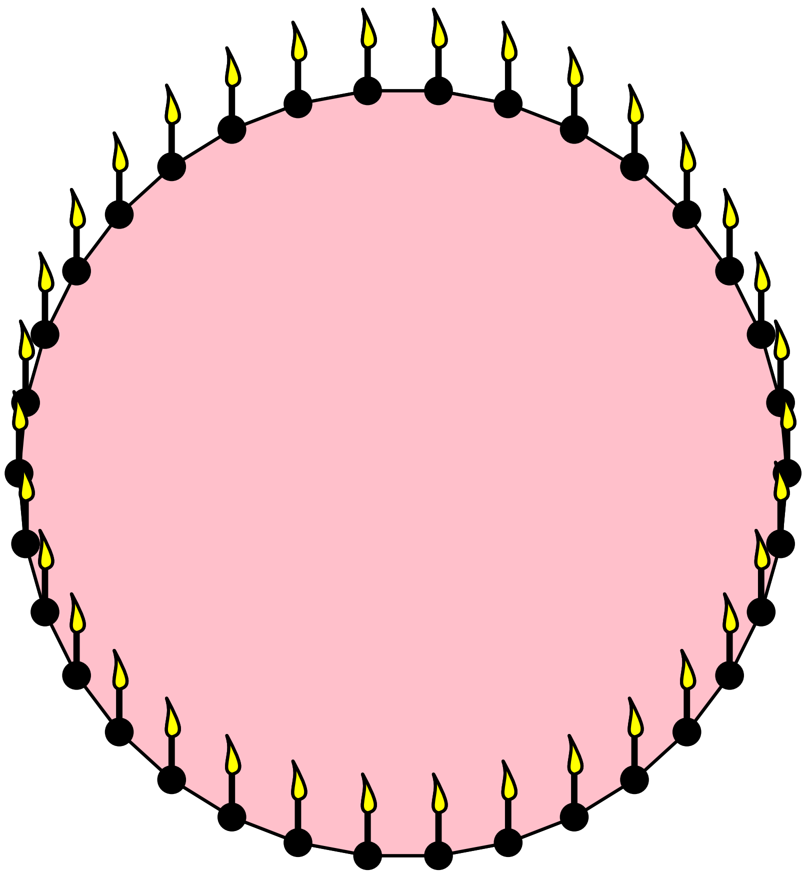

Counting problem

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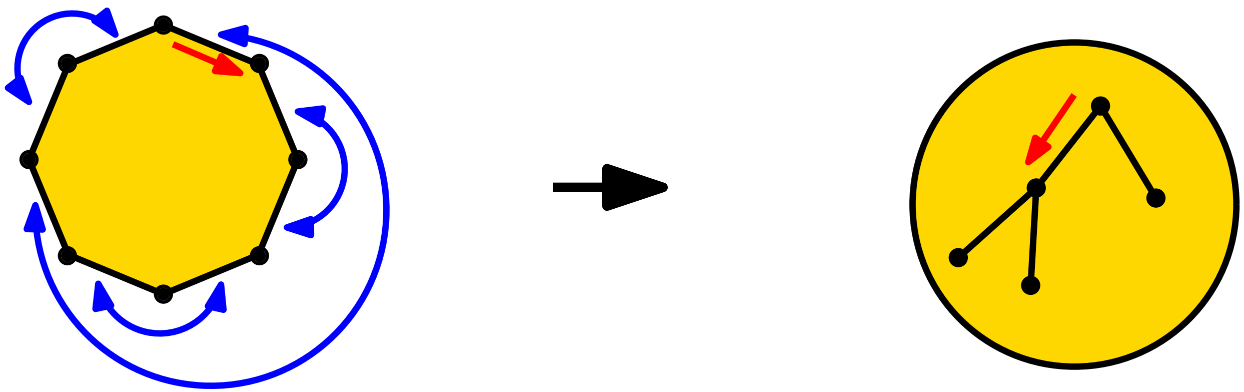
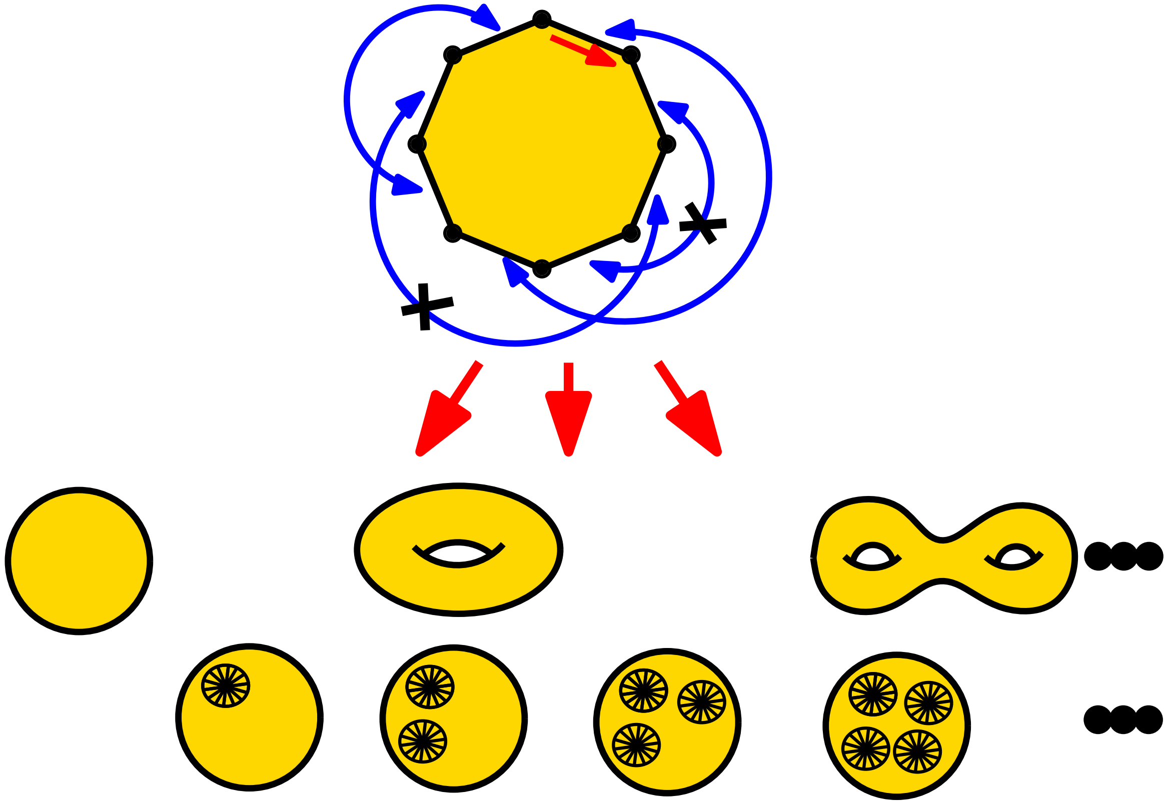
Remark 1. The surface obtained is characterized by its orientability and the number of vertices of the one-face map.

Remark 2. The number of ways of getting the sphere is the Catalan number $\mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}$.

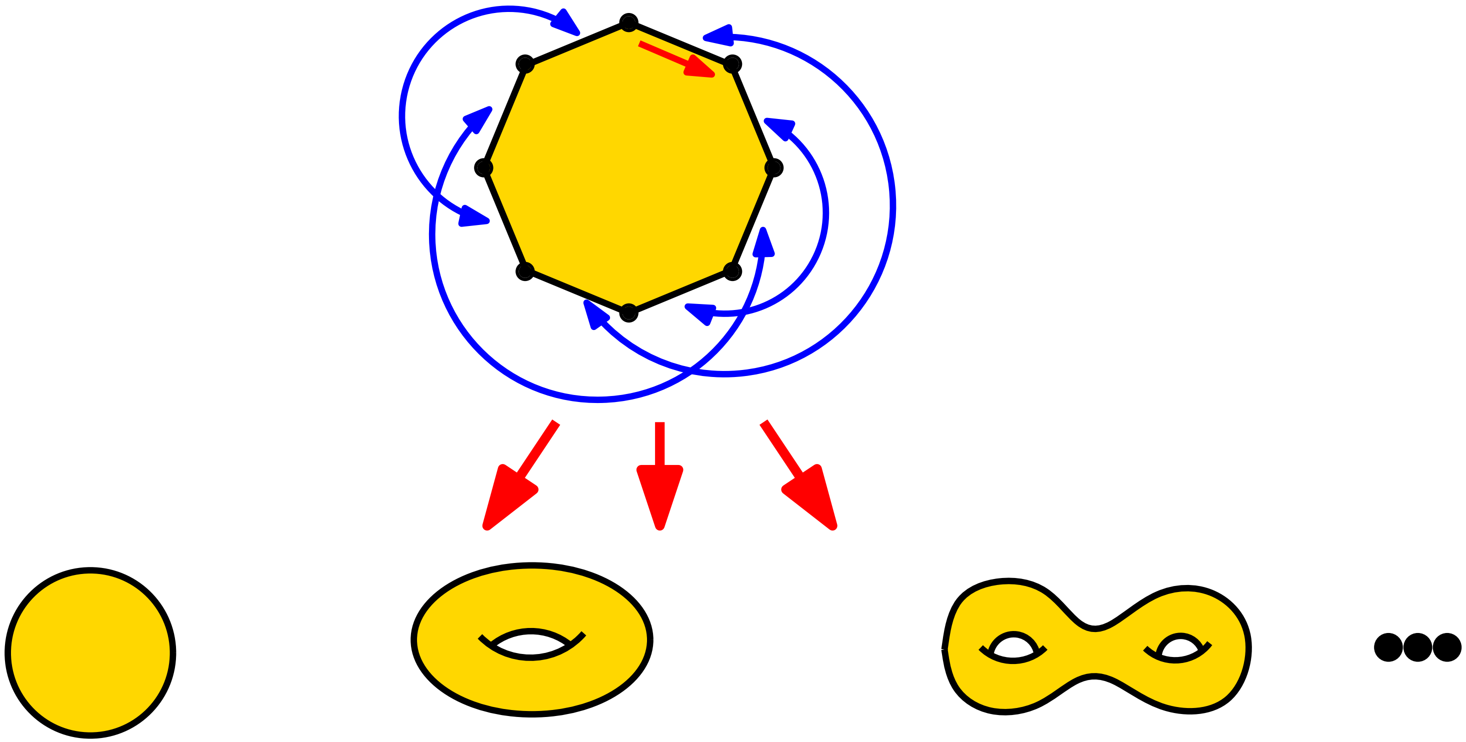


Results



Colored gluings

Question: What is the number of one-face maps on orientable surfaces with n edges and v vertices ?



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Theorem [Harer, Zagier 86].

$$\sum_{\text{orientable one-face maps}} p^{\#\text{vertices}} = \sum_{q=1}^{p} \binom{p}{q} 2^{q-1} \binom{n}{q-1} (2n-1)!!$$

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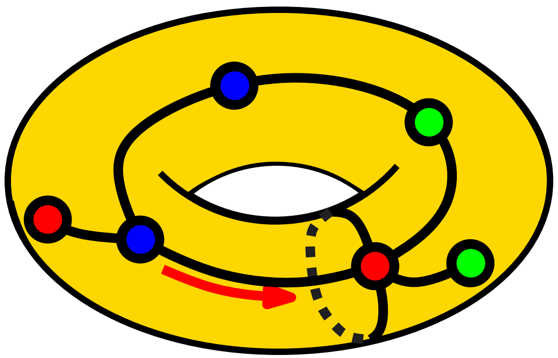
Combinatorial interpretation: the number of orientable one-face maps with vertices colored using all the colors in $[q] := \{1, 2, \ldots, q\}$ is

$$2^{q-1}\binom{n}{q-1}(2n-1)!!.$$



Results

Theorem [B.]: The number of one-face maps with n edges and vertices colored using every color in $[q]$ is

$$\sum_{r=1}^{n-q+2} \frac{q!r!}{2^{r-1}} P_{q,r} \binom{2n}{2q+2r-4} (2n-2q-2r+1)!!$$

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where $P_{q,r}$ is the number of planar maps with q vertices and r faces.

Remark. $P_{q,r}$ is the coefficient of $x^q y^r$ in the series P defined by:

$$27P^4 - (36x+36y-1)P^3$$
$$+(24x^2y+24xy^2-16x^3-16y^3+8x^2+8y^2+46xy-x-y)P^2$$
$$+xy(16x^2+16y^2-64xy-8x-8y+1)P$$
$$-x^2y^2(16x^2+16y^2-32xy-8x-8y+1)=0.$$

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Corollary [Ledoux 09] The number $\mu_v(n)$ of one-face maps with n edges and v vertices satisfies

$$\begin{aligned}(n+1)\,\eta_v(n) = &(4n-1)\,(2\,\eta_{v-1}(n-1) - \eta_v(n-1)) \\ &+(2n-3)\,((10n^2-9n)\,\eta_v(n-2) + 8\,\eta_{v-1}(n-2) - 8\,\eta_{v-2}(n-2)) \\ &+5(2n-3)(2n-4)(2n-5)\,(\eta_v(n-3) - 2\,\eta_{v-1}(n-3)) \\ &-2(2n-3)(2n-4)(2n-5)(2n-6)(2n-7)\,\eta_v(n-4).\end{aligned}$$

Sketch of proof:

Recurrence $\longleftrightarrow$ differential equation for $F(x,z) = \sum_{n,v} \eta_v(n) \frac{x^v z^n}{(2n)!}$

$\longleftrightarrow$ differential equation for $G(x,z) = \sum_{n,q} C_{n,q} \frac{x^q z^n}{(2n)!}$

$\longleftrightarrow$ differential equation for $P(x,y) = \sum_{q,r} P_{q,r} x^q y^r$. $\square$

Results

Theorem [B.]: The number of one-face maps with n edges and vertices colored using every color in $[q]$ is

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Other known formulas:

Theorem [Goulden, Jackson 97]

$$\sum_{\substack{\text{one-face map} \\ \text{with } n \text{ edges}}} p^{\#vertices} = p\, n! \sum_{k=0}^{n} 2^{2n-k} \sum_{r=0}^{n} \binom{n-\frac{1}{2}}{n-r} \binom{k+r-1}{k} \binom{\frac{p-1}{2}}{r}$$

$$+ p\,(2n-1)!! \sum_{q=1}^{p-1} 2^{q-1} \binom{p-1}{q} \binom{n}{q-1}.$$

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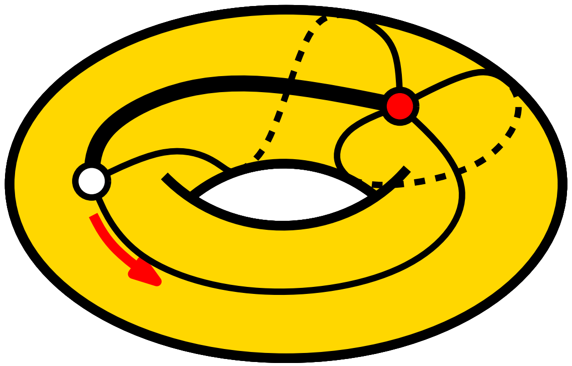
Theorem [B., Chapuy 10] $\eta_v(n) \underset{n\to\infty}{\sim} c_{n-v+1} n^{3(n-v)/2} 4^n$,

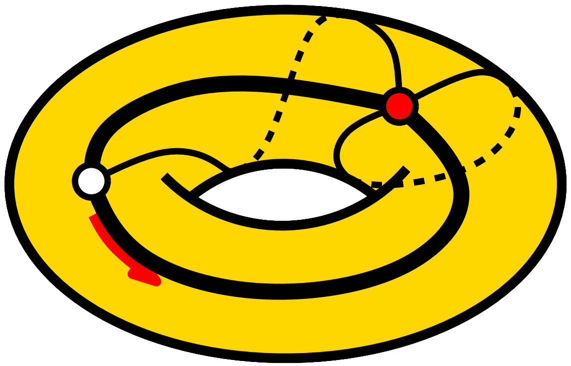
where $c_t = \begin{cases} \frac{2^{t-2}}{\sqrt{6}^{t-1}(t-1)!!} & \text{if } t \text{ odd}, \\ \frac{3\cdot 2^{t-2}}{\sqrt{\pi}\sqrt{6}^{t}(t-1)!!} \sum_{i=1}^{t/2-1} \binom{2i}{i} 16^{-i} & \text{if } t \text{ even}. \end{cases}$

Results: Bijections

A **tree-rooted map** is a map on an orientable surface with a marked spanning tree.

A **planar-rooted map** is a map on an orientable surface with a marked planar connected spanning submap.





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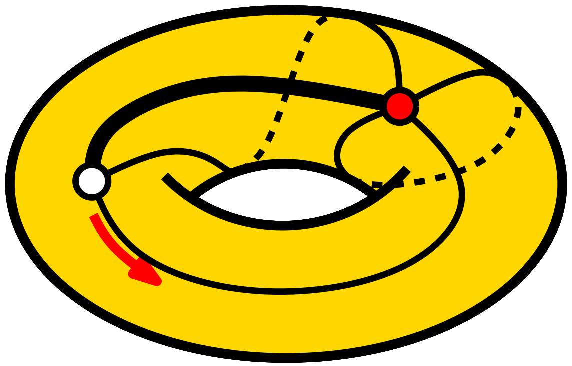
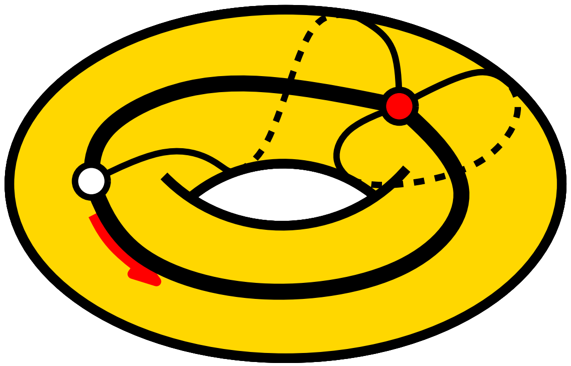




The number of tree-rooted maps with q vertices and n edges is

$$\mathrm{Cat}(q-1)\binom{2n}{2q}(2n-2q+1)!! = \frac{2^{q-1}}{q!}\binom{n}{q-1}(2n-1)!!$$

The number of planar-rooted maps with q vertices, r faces, and n edges is $P_{q,r}\binom{2n}{2q+2r-4}(2n-2q-2r+1)!!$

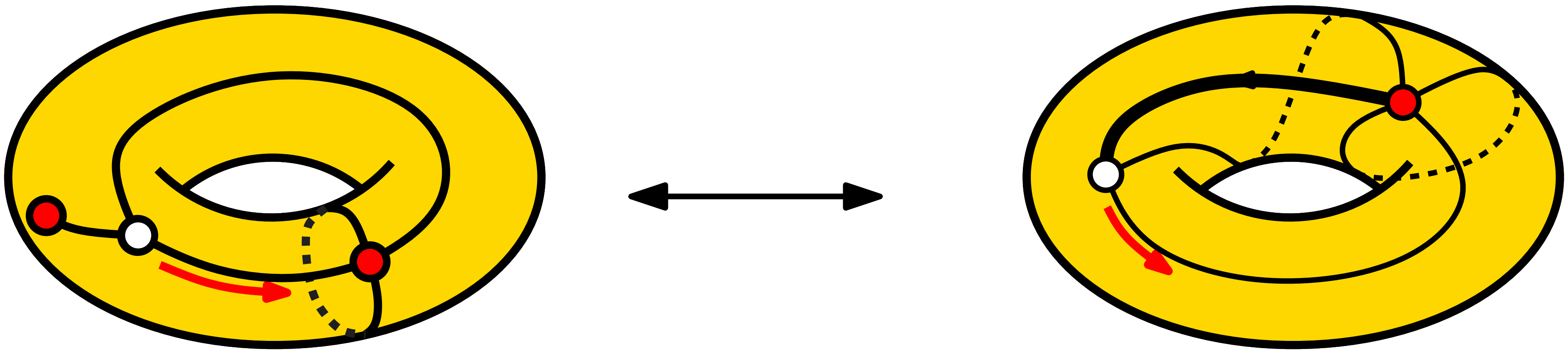
Results: Bijections

Thm [Bernardi - Inspired by Lass] Bijection between

• **one-face maps on orientable surface** with n edges and vertices colored using every color in $[q]$

• **tree-rooted maps** with n edges and q labeled vertices.

$\#$ edges between colors i and j $\leftrightarrow$ $\#$ edges between vertices i and j.



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$\#$ edges between colors i and j $\leftrightarrow$ $\#$ edges between vertices i and j.

Corollary 1. [Harer-Zagier 86, Lass 01, Goulden, Nica 05] The number of $[q]$-colored orientable one-face maps with n edges is

$$2^{q-1}\binom{n}{q-1}(2n-1)!!$$

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Refinement. [B.] The number of such map with color degrees $\alpha_1, \ldots, \alpha_q$ is $2^{q-n}n\dfrac{(2n-q)!}{(n-q+1)!}$.

Proof. Same number for each of the $\binom{2n-1}{q-1}$ possible color degrees . $\square$

Results: Bijections

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Corollary 2 [Jackson 88, Schaeffer ,Vassilieva 08] The number of bipartite $[q],[r]$-colored orientable one-face maps with n edges is

$$n!\binom{n-1}{q-1, r-1, n-q-r+1}.$$

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Refinement. [Morales, Vassilieva 09] The number of such map with color degrees $\alpha_1,\ldots,\alpha_q,\beta_1,\ldots,\beta_r$ is $\dfrac{n\,(n-q)!(n-r)!}{(n-q-r+1)!}$.

Results: Bijections

Thm [B.] There is a $q!r!2^{1-r}$-to-1 correspondence between

• **one-face maps on general surfaces** with n edges and vertices colored using every color in $[q]$

• **planar-rooted maps** with with n edges, q vertices, and r faces.

Moreover, $\#$ edges incident to color $i \leftrightarrow$ #edges incident to vertex i.

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Moreover, # edges incident to color $i \leftrightarrow$ #edges incident to vertex i.

Corollary: The number of $[q]$-colored rooted one-face maps with n edges on general surfaces is:

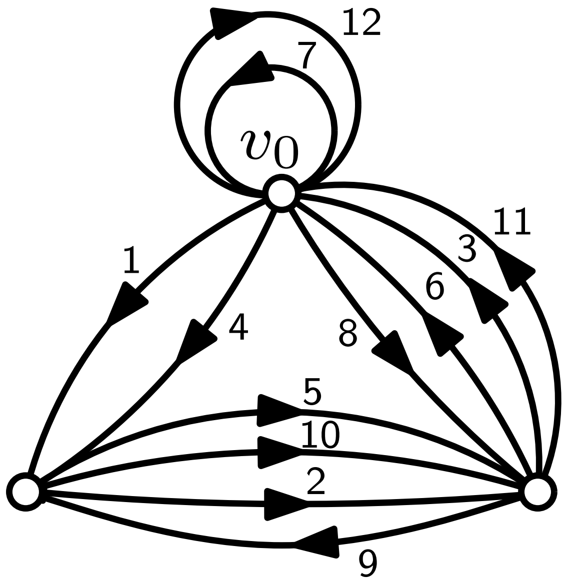
$$\sum_{r=1}^{n-q+2} \frac{q!r!}{2^{r-1}} P_{q,r} \binom{2n}{2q+2r-4} (2n-2q-2r+1)!!$$

where $P_{q,r}$ is the number of planar maps with q vertices and r faces.

Sketch of proof - Orientable case

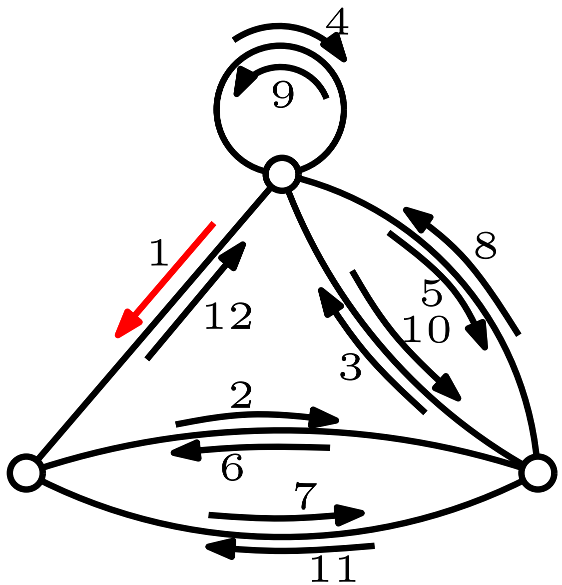
Idea 1: Colored unicellular map $\leftrightarrow$ Eulerian tour

Def. An **Eulerian tour** of a directed graph is a walk starting and ending at the same vertex and using every arc exactly once.



Idea 1: Colored unicellular map ↔ Eulerian tour

Def. An **Eulerian tour** of an **undirected graph** is a walk starting and ending at the same vertex and using every direction of every edge exactly once.



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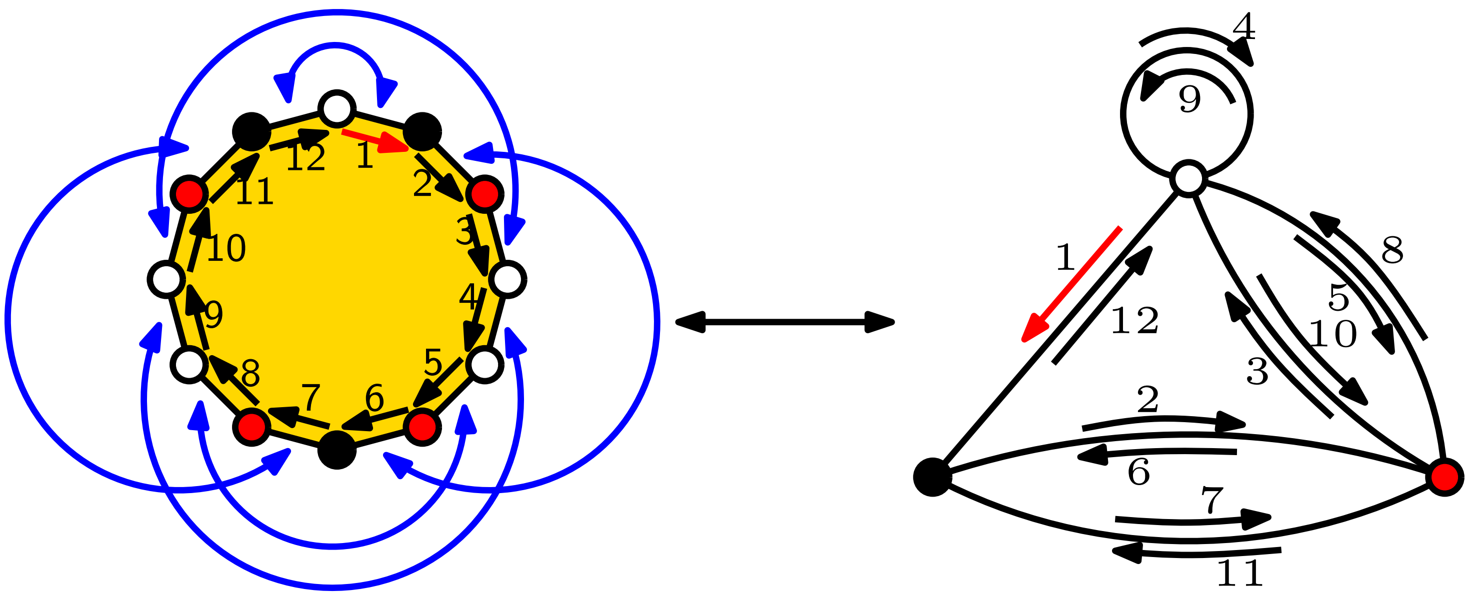
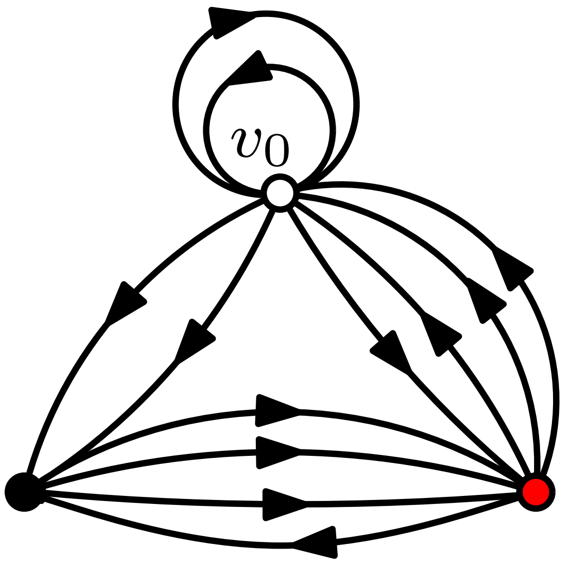


Lemma [Lass 01]. Bijection between $[q]$-colored one-face maps and set of pairs (G, E), where G is an undirected graph with vertex set $[q]$ and E is an Eulerian tour.

Idea 2: BEST Theorem

BEST Theorem. (de Bruijn, van Aardenne-Ehrenfest, Smith and Tutte)
Fix $\vec{G}$ digraph with as many ingoing and outgoing arcs at each vertex. The Eulerian tours of $\vec{G}$ starting and ending at v_0 are in bijection with pairs (T, R), where T is a spanning tree oriented toward v_0 and R is an ordering around each vertex of the outgoing arc not in T.



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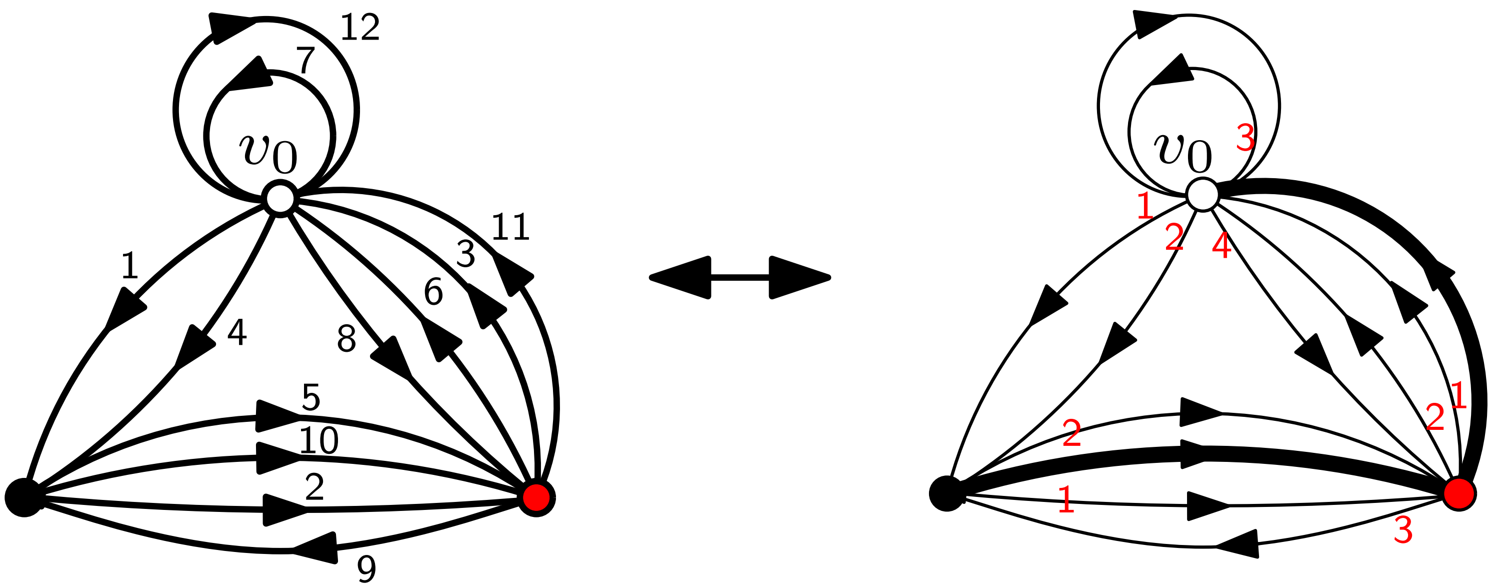

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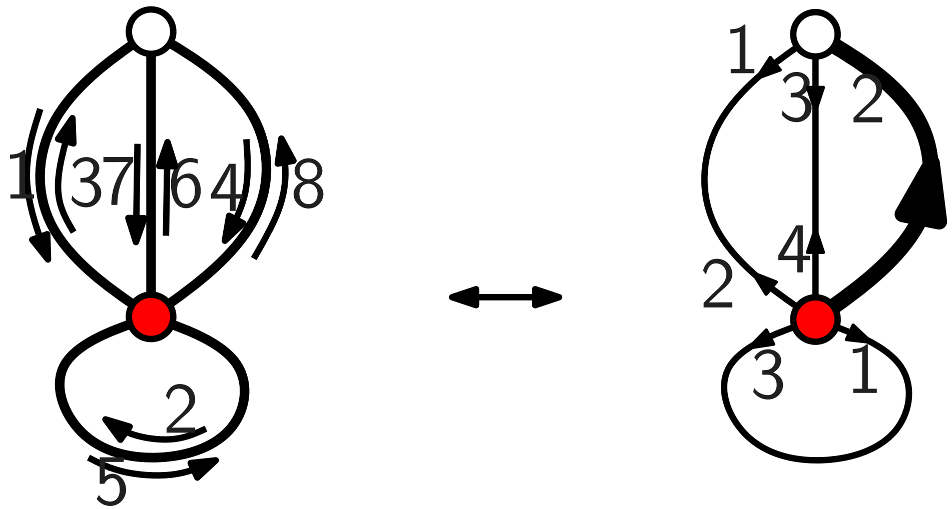
Corollary. The Eulerian tours of an undirected graph G are in bijection with pairs (T, R), where
- T is a spanning tree (rooted as v_0)
- R is an order at each vertex v, of all the edges incident to v except the parent edge of v in T.



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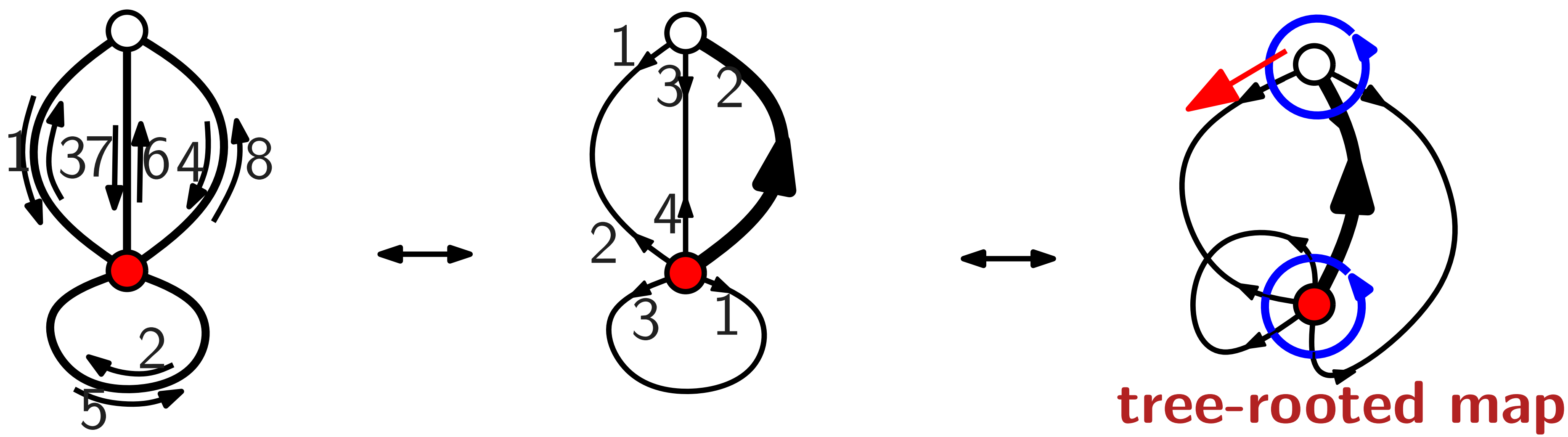
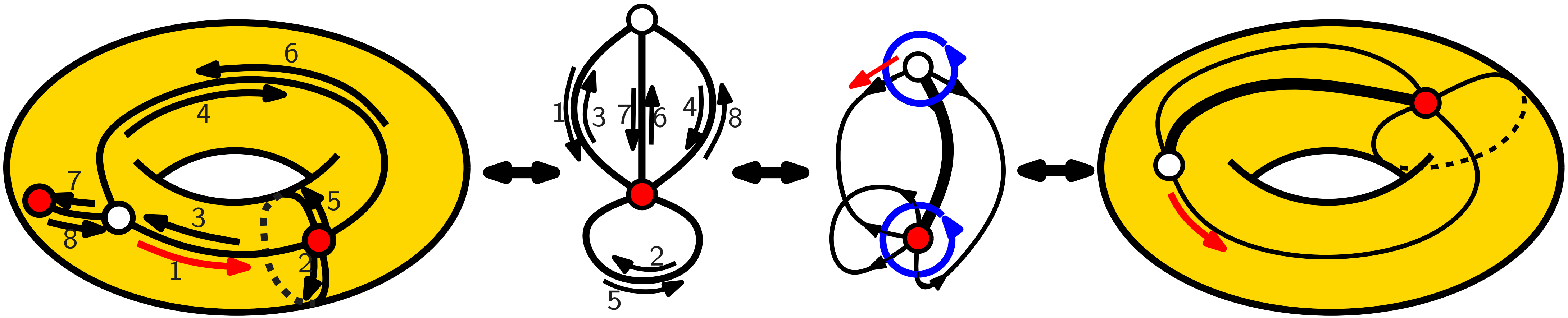
Corollary. The Eulerian tours of an undirected graph G are in bijection with pairs (T, R), where
- T a spanning tree + a marked half-edge at v_0,
- R a cyclic ordering of the half-edges at each vertex.



Summary for orientable gluings

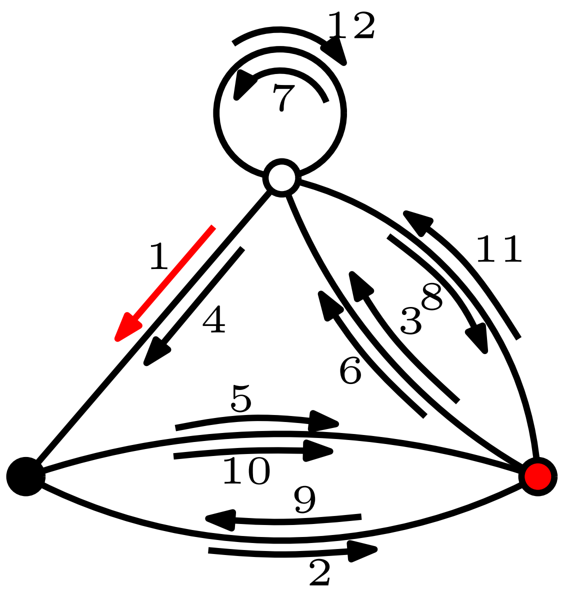
Summary: **Bijection** between rooted one-face maps colored using every color in $[q]$ with n edges on orientable surfaces and tree-rooted maps with q labelled vertices and n edges.



Sketch of proof - general case

Idea 1: Colored unicellular map $\leftrightarrow$ bi-Eulerian tour

Def. A **bi-Eulerian tour** of an undirected graph is a walk starting and ending at the same vertex and using every edge twice.



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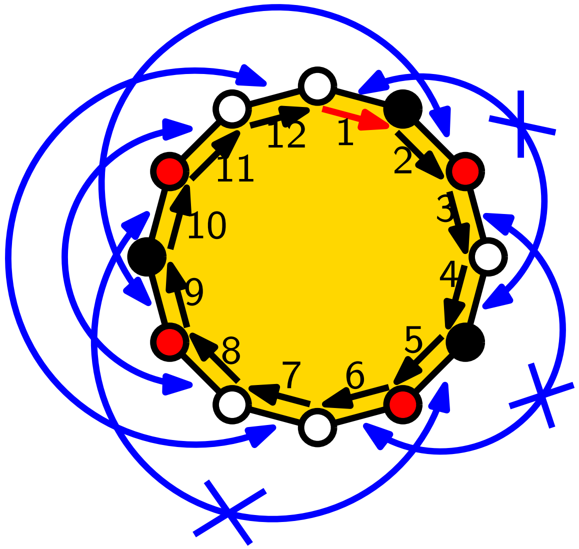
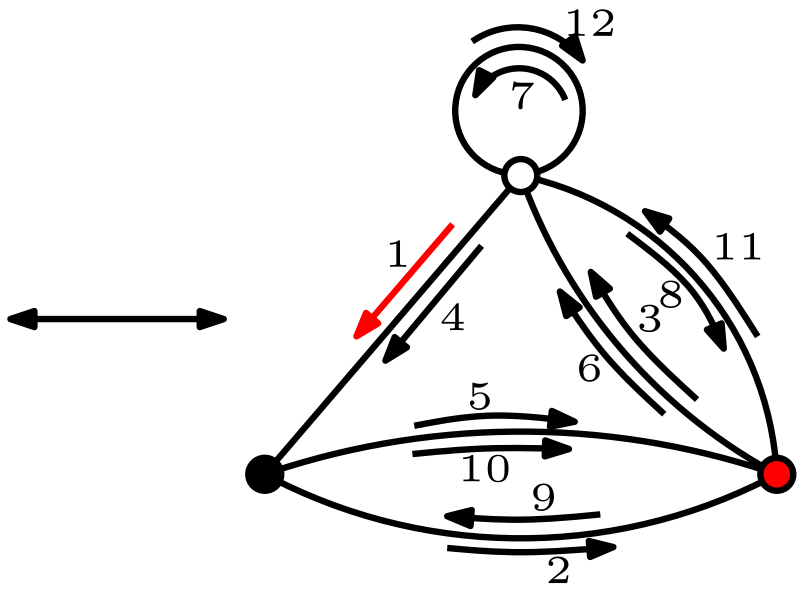




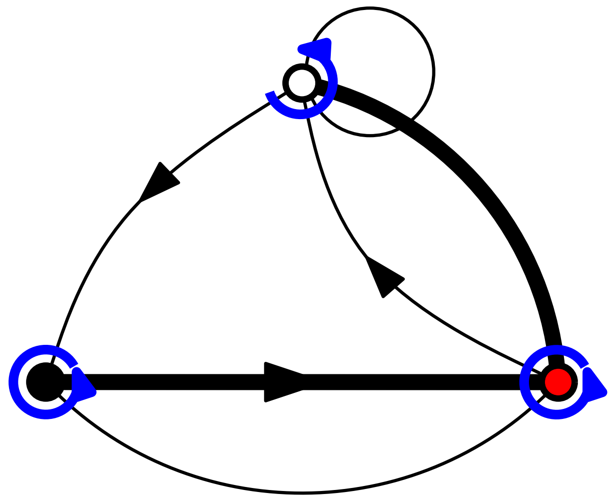
Lemma [adapting Lass 01]. Bijection between $[q]$-colored one-face maps and set of pairs (G, E), where G is a graph with vertex set $[q]$ and E is a bi-Eulerian tour.

Moreover, the map is on an orientable surface if and only if no edge is used twice in the same direction.

Idea 2: BEST Theorem (adapted to general situation)

A **bi-oriented tree-rooted map** is a rooted map on orientable surface + spanning tree + partial orientation such that

- indegree=outdegree for every vertex
- oriented edges in the spanning tree are oriented toward parent.



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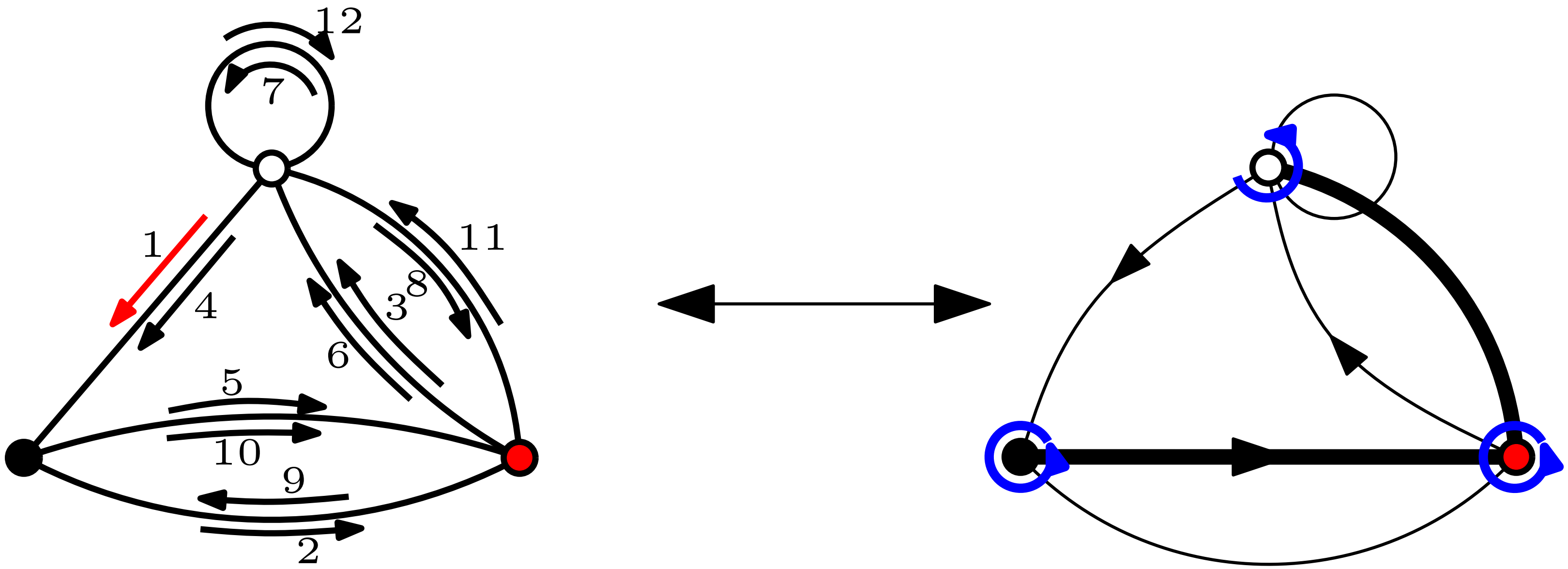
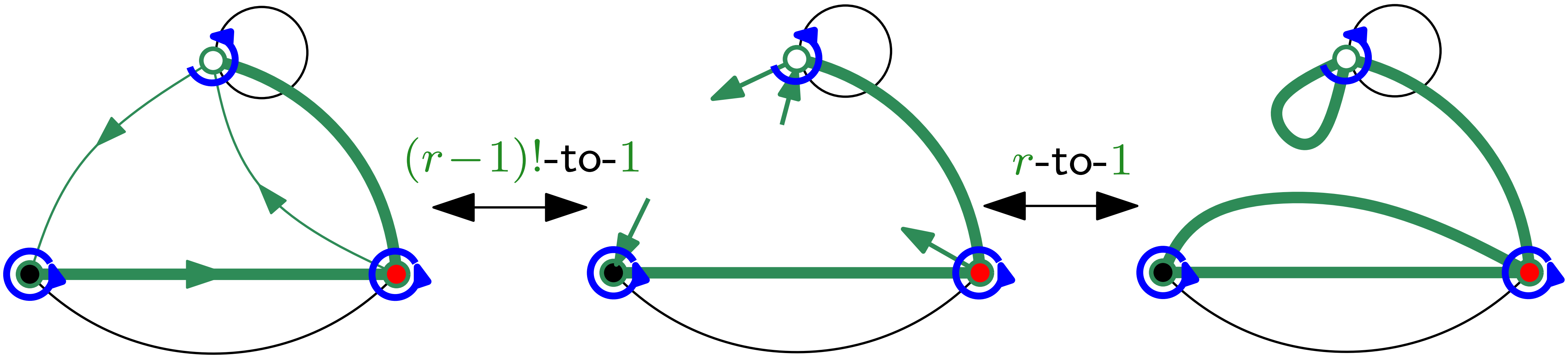
- indegree=outdegree for every vertex
- oriented edges in the spanning tree are oriented toward parent.

Corollary of BEST. Let G be an undirected graph.
There is a 1-to-2^r correspondence between the bi-Eulerian tours of G and the bi-oriented tree-rooted map on G with r oriented edges outside of the spanning tree.



Idea 3: cutting and pasting

Def. Let B be a bi-oriented tree-rooted map. The planar-rooted map $P = \Psi(B)$ is obtained by cutting the oriented external edges in their middle and regluing them according to the parenthesis system they form around the tree (and then forgetting the tree + orientation).



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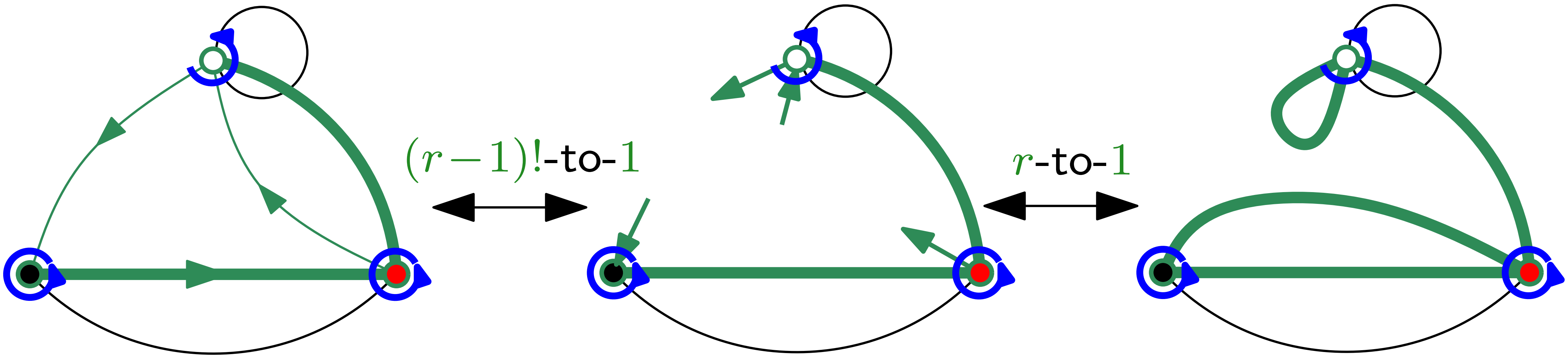
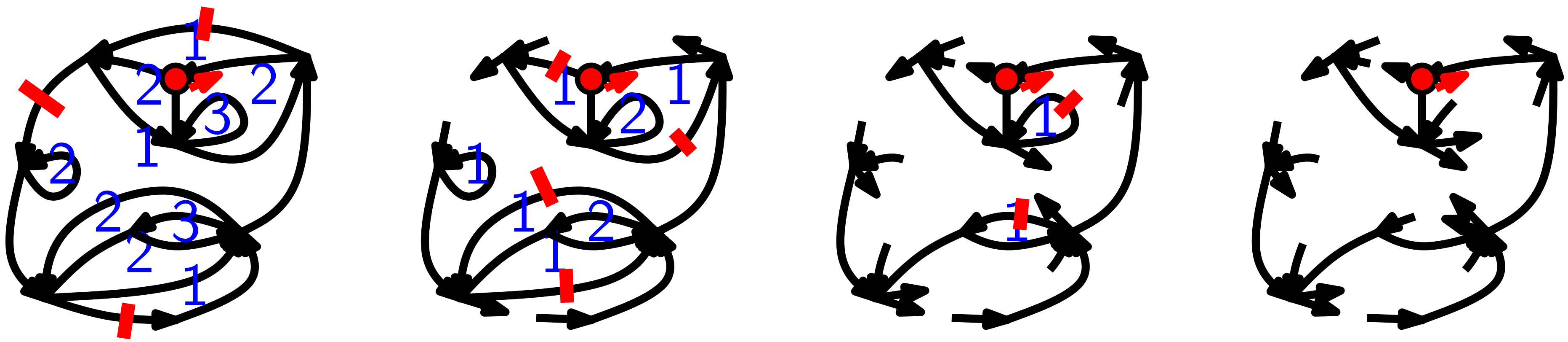


Theorem: The mapping Ψ is $r!$-to-1 between bi-oriented tree-rooted maps with $r-1$ oriented edges outside of the spanning tree and planar-rooted map with r sub-faces.

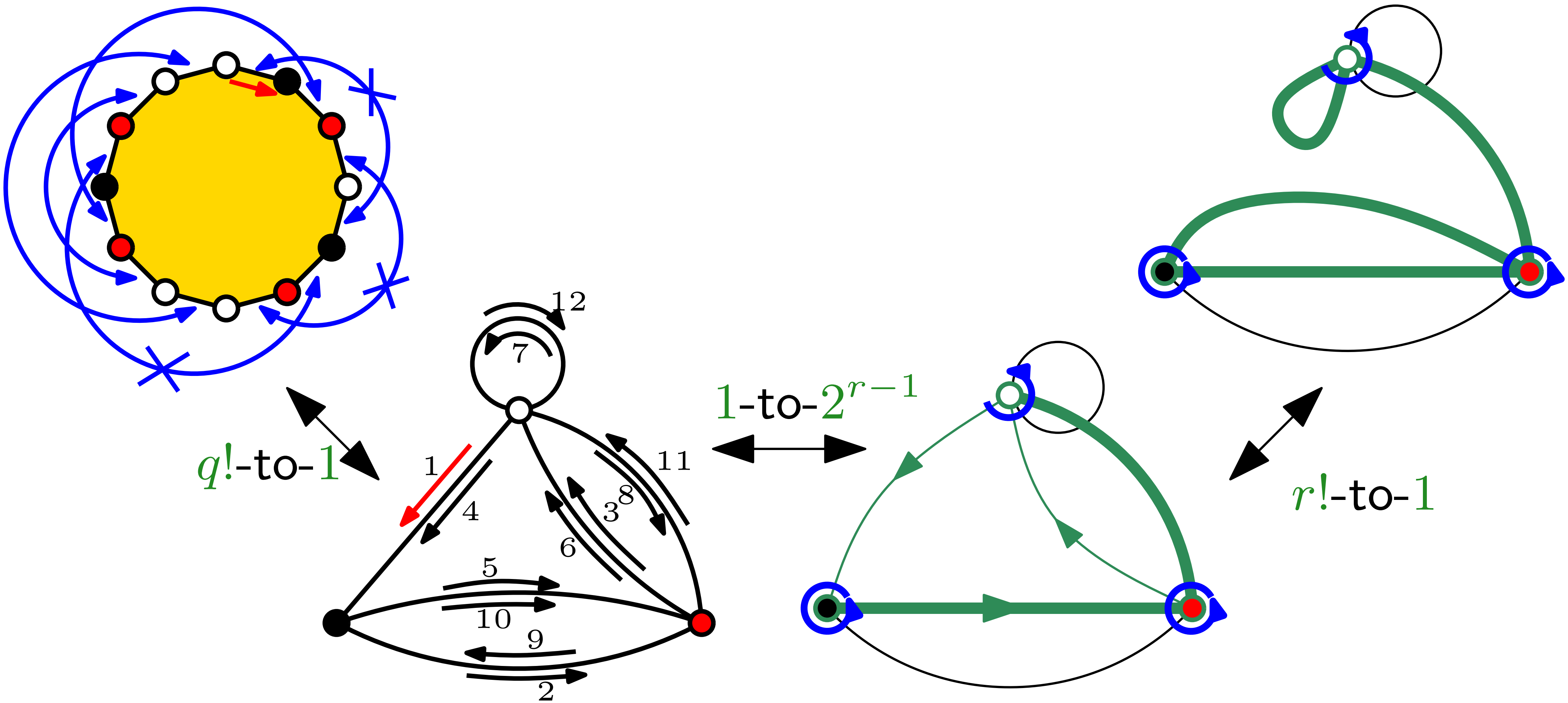
Idea 3: cutting and pasting

Underlying thm (related to [Bouttier, Di Francesco, Guitter 02]): Bijection between planar maps with a marked face and partially oriented plane-tree with additional oriented half-edges such that indegree=outdegree at each vertex.



Summary for general gluings

Summary: There is a $q!r!2^{r-1}$-to-1 correspondence between rooted one-face maps colored using every color in $[q]$ with n edges on general surfaces and planar-rooted maps with q vertices, r faces, and n edges.



One-face constellations

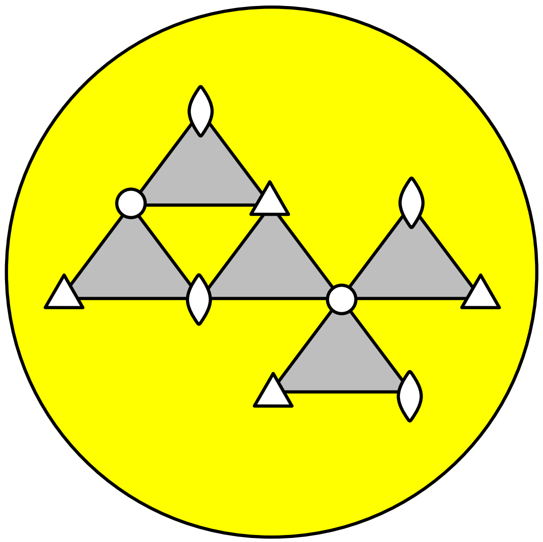
Joint work with Alejandro Morales

Constellations (= drawings of factorizations)

A k-**constellation** is a map on an orientable surface with black-white coloring of faces, in which vertices have a *type* in $\{1, \ldots, k\}$, such that every black faces has degree k, with vertices of type $1, 2, \ldots, k$ clockwise.

Example of 3-constellation:

○ Type 1
◊ Type 2
△ Type 3



Constellations (= drawings of factorizations)

Relation with permutations:

k-constellations with n labelled black faces	$\longleftrightarrow$	Tuples $(\pi_1, \ldots, \pi_k)$ of permutations acting transitively on $\{1, 2, \ldots, n\}$
Vertex of type t	$\longleftrightarrow$	Cycle of π_t
White faces	$\longleftrightarrow$	Cycle of product $\pi_1 \pi_2 \cdots \pi_k$

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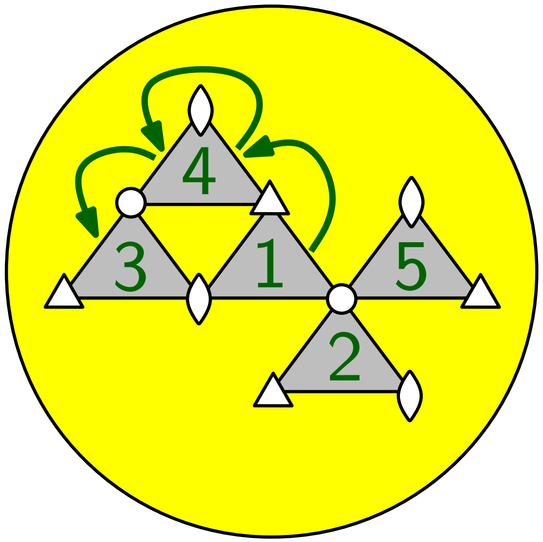

$$\pi_1 = (1, 2, 5)(3, 4)$$
$$\pi_2 = (1, 3)(2)(4)(5)$$
$$\pi_3 = (1, 4)(2)(3)(5)$$
$$\pi_1 \pi_2 \pi_3 = (1, 3, 2, 5)(4)$$

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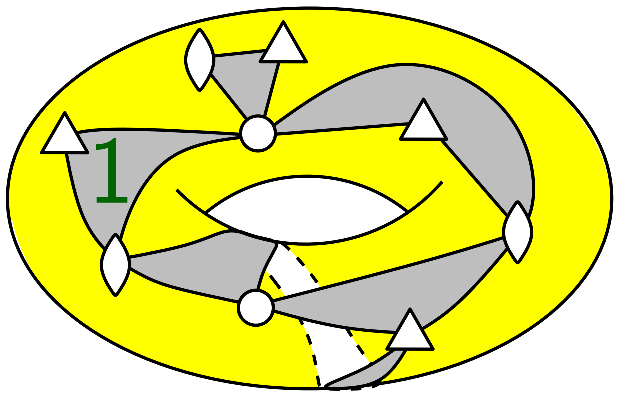
Conclusion for one-face constellations:

one-face k-constellations with a marked black face	$\longleftrightarrow$	Tuples $(\pi_1, \ldots, \pi_k)$ such that $\pi_1 \pi_2 \cdots \pi_k = (1, 2, \ldots, k)$.



Jackson formula

Question: Let $\lambda_1, \ldots, \lambda_k$ be partitions of n.
How many factorizations $\pi_1\pi_2\ldots\pi_k = (1, 2, \ldots, n)$ are there with permutation π_t of cycle type λ_t?
$\longleftrightarrow$ How many one-face k-constellations with vertices of type t having degrees given by λ_t?

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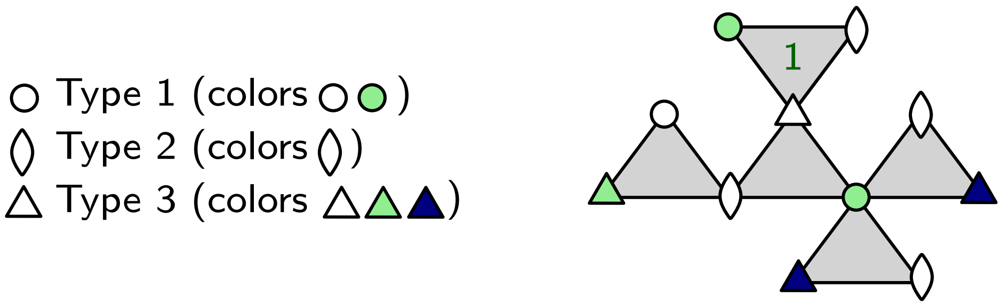
Reformulation with colors.
As before, the nice formulas are for **vertex-colored constellations**.



Equivalent question: Let $\lambda_1, \ldots, \lambda_k$ be partitions of n.
How many vertex-colored one-face k-constellations with vertices of type t having color degrees λ_t?

Jackson formula

Theorem [Jackson 88] Let $q_1, \ldots, q_k > 0$.
The number of vertex-colored one-face k-constellations with vertices of type t using all the colors in $[q_t]$ is

$$n!^{k-1} [x_1^{q_1-1} \cdots x_k^{q_k-1}] \left(\prod_{t=1}^{k} (1 + x_t) - \prod_{t=1}^{k} x_t \right)^{n-1}.$$

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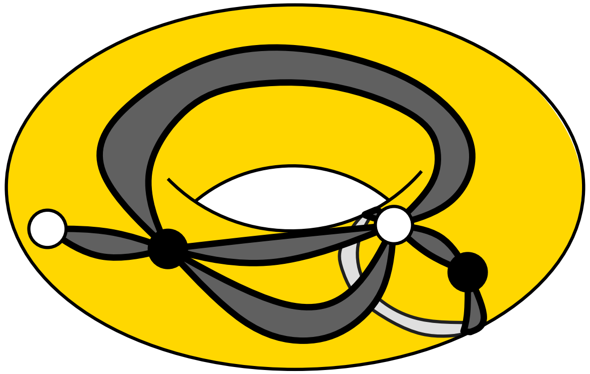
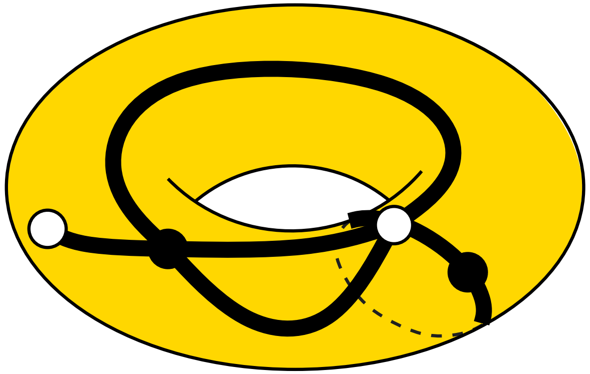
$\#\{(S_1, \ldots, S_{n-1}) |\ S_i \subsetneq [n]$, and $\forall t \in [k],\ t$ appears in $q_k - 1$ subsets$\}$

Bijective proof?

Case $k = 2$

We have already solved the case $k = 2$!
Indeed one-face 2-constellations identify with one-face bipartite maps.





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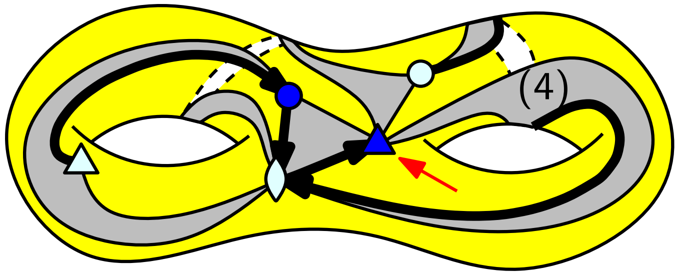
Theorem [Jackson 88, Schaeffer ,Vassilieva 08] The number of bipartite $[q], [r]$-colored orientable one-face maps with n edges is

$$n!\binom{n-1}{q-1, r-1, n-q-r+1}.$$

Refinement. [Morales, Vassilieva 09] The number of such map with color degrees $\alpha_1, \ldots, \alpha_q, \beta_1, \ldots, \beta_r$ is $\dfrac{n\,(n-q)!(n-r)!}{(n-q-r+1)!}$.

General $k \geq 2$

A **tree-rooted k-constellation** is a rooted k-constellation with a marked spanning tree such that the type of each vertex is the type of its parent - 1.



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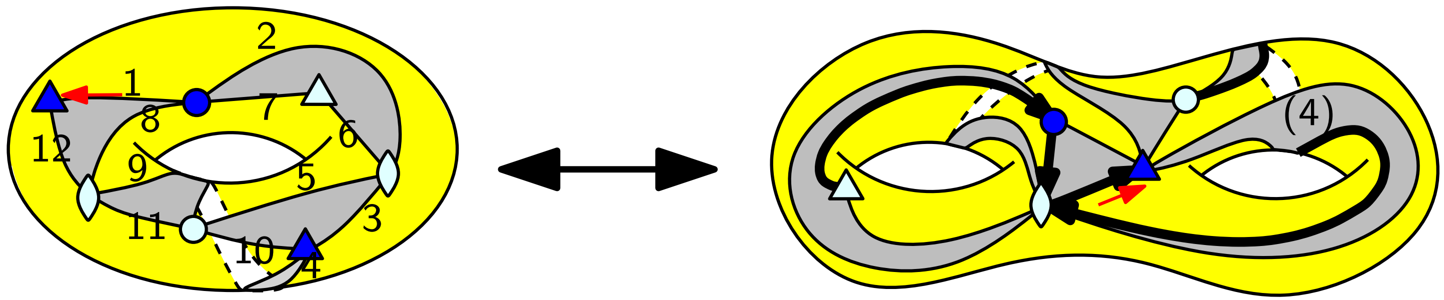
Theorem [B., Morales] One-face k-constellations with vertices colored using colors $[q_1], [q_2], \dots, [q_k]$ are in bijection with tree-rooted k-constellations with q_t labelled vertices of type t.
Moreover, the color degree in one-face constellation = degree in tree-rooted constellation.



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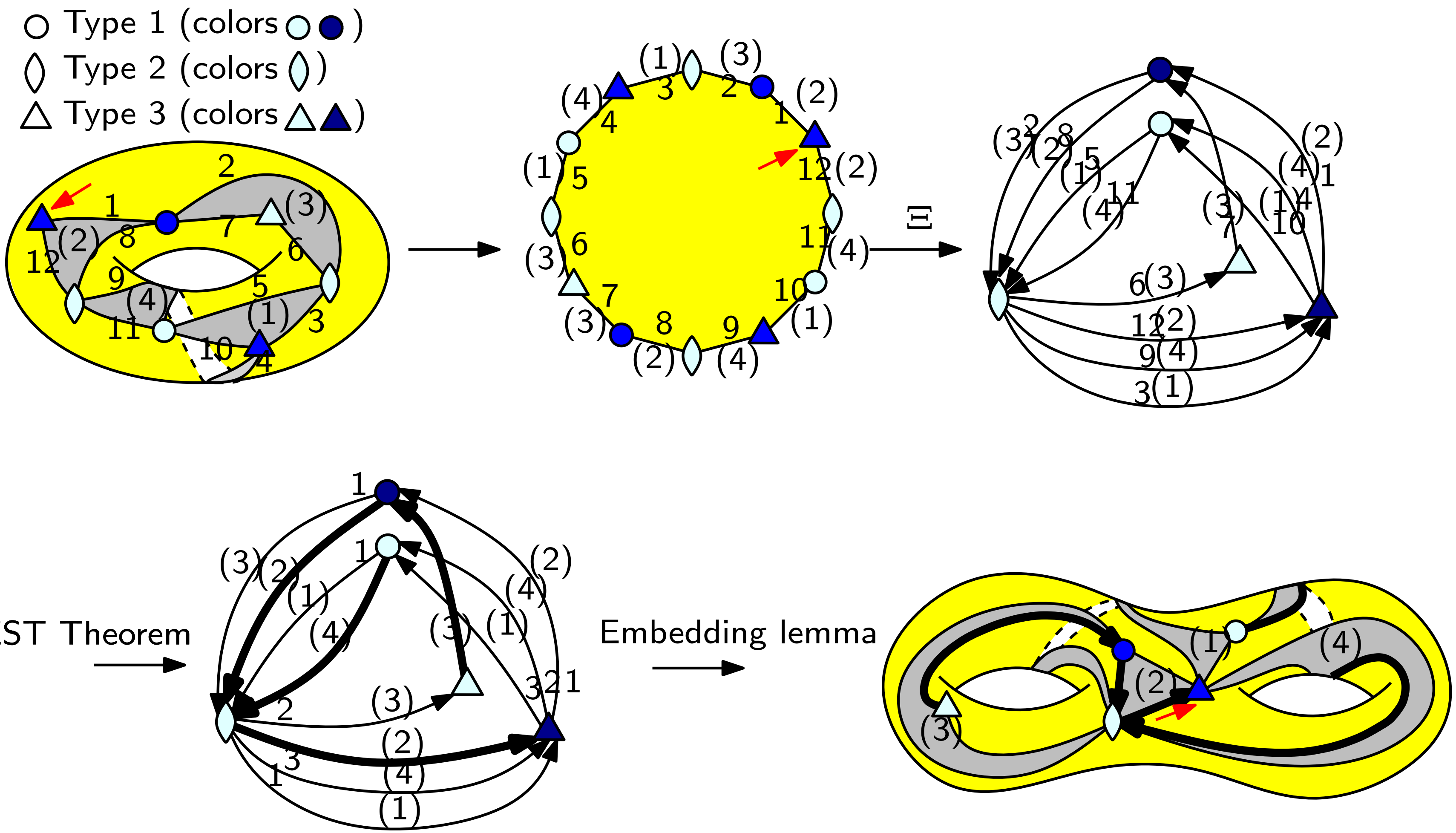
Moreover, the color degree in one-face constellation $=$ degree in tree-rooted constellation.

Corollary [B., Morales] The number of vertex-colored k-constellations with color degree $\lambda_1, \ldots, \lambda_k$ only depends on $\ell(\lambda_1), \ldots, \ell(\lambda_k)$.

Sketch of proof:



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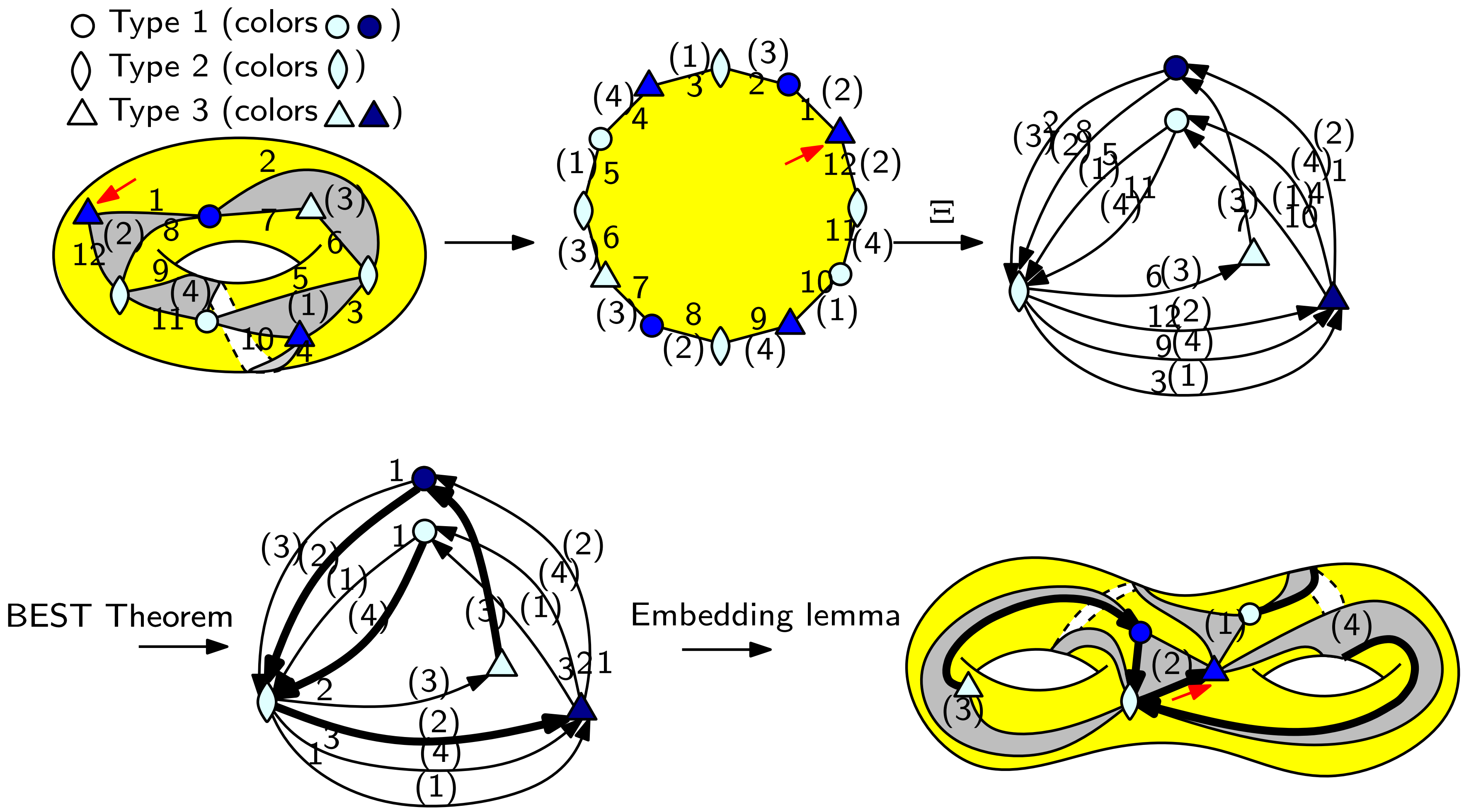

Related construction [Vassilieva 2014?].

Relation with Jackson formula?

Not easy to see that tree-rooted k-constellations with n edges and q_t labelled vertices of type t are counted by

$$n!^{k-1}[x_1^{q_1-1}\cdots x_k^{q_k-1}]\left(\prod_{t=1}^{k}(1+x_t)-\prod_{t=1}^{k}x_t\right)^{n-1}.$$

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For instance, recursive decomposition of tree-rooted constellation+ Lagrange inversion gives an expression which is more complicated than Jackson's formula [Vassilieva 2014?]

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Additional ideas: the dual of tree rooted maps are some kind of one-face maps, and one can reuse the BEST theorem again. . .
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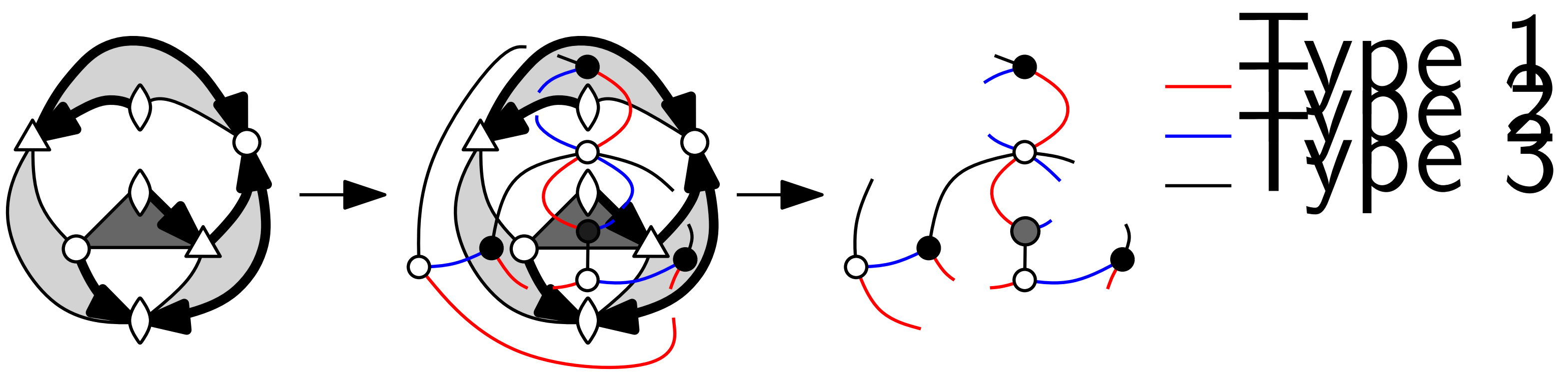

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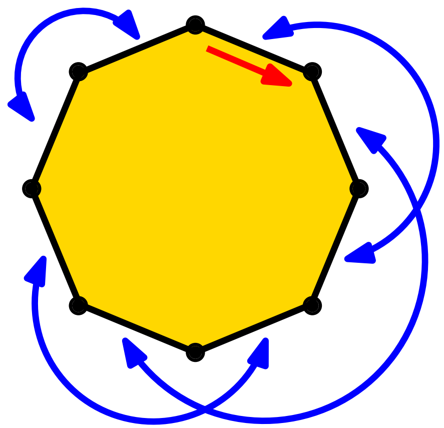
Biddings are easier to count ... but still not exactly Jackson formula.
$\rightsquigarrow$ Probabilistic puzzle (solved [B., Morales]).

Thanks.



Probabilistic puzzle:

Theorem: Let $S_1, \ldots, S_{k-1} \subsetneq [k]$ with t appearing q_t times. Probability of getting a tree from the rule below is the same as proba that $S_1 = \emptyset$.