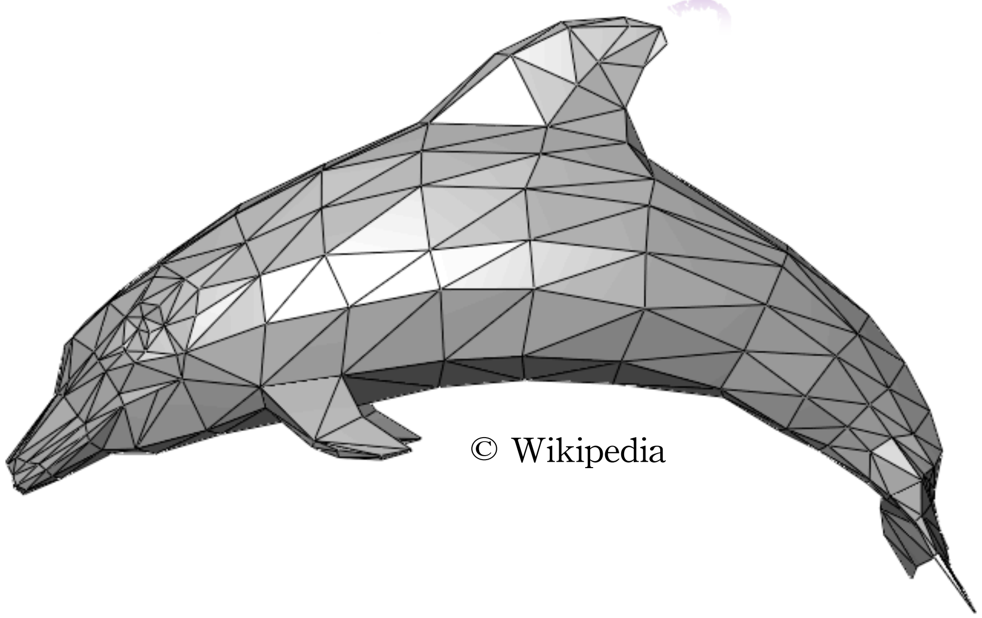
# Topological recursion: computing asymptotics of knot invariants like counting maps ?



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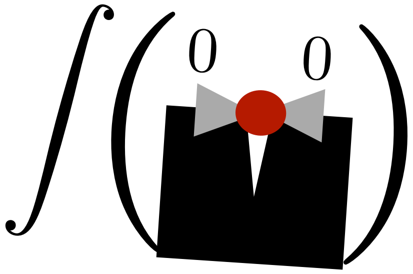
Journées Cartes
IPhT, 21 juin 2013

# Topological recursion: computing asymptotics of knot invariants like counting maps ?

1 Maps with(out) tubes ...

2 ... and formal matrix models



3 Introduction to knot invariants W(G,R)

4 Torus knots and W(G,R)

5 Hyperbolic knots and W(G,R)

# 1 Maps with(out) tubes ...

Definition

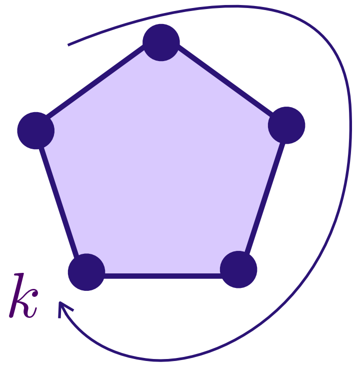
Planar maps and substitution

Planar 1-cut lemma

Higher topologies

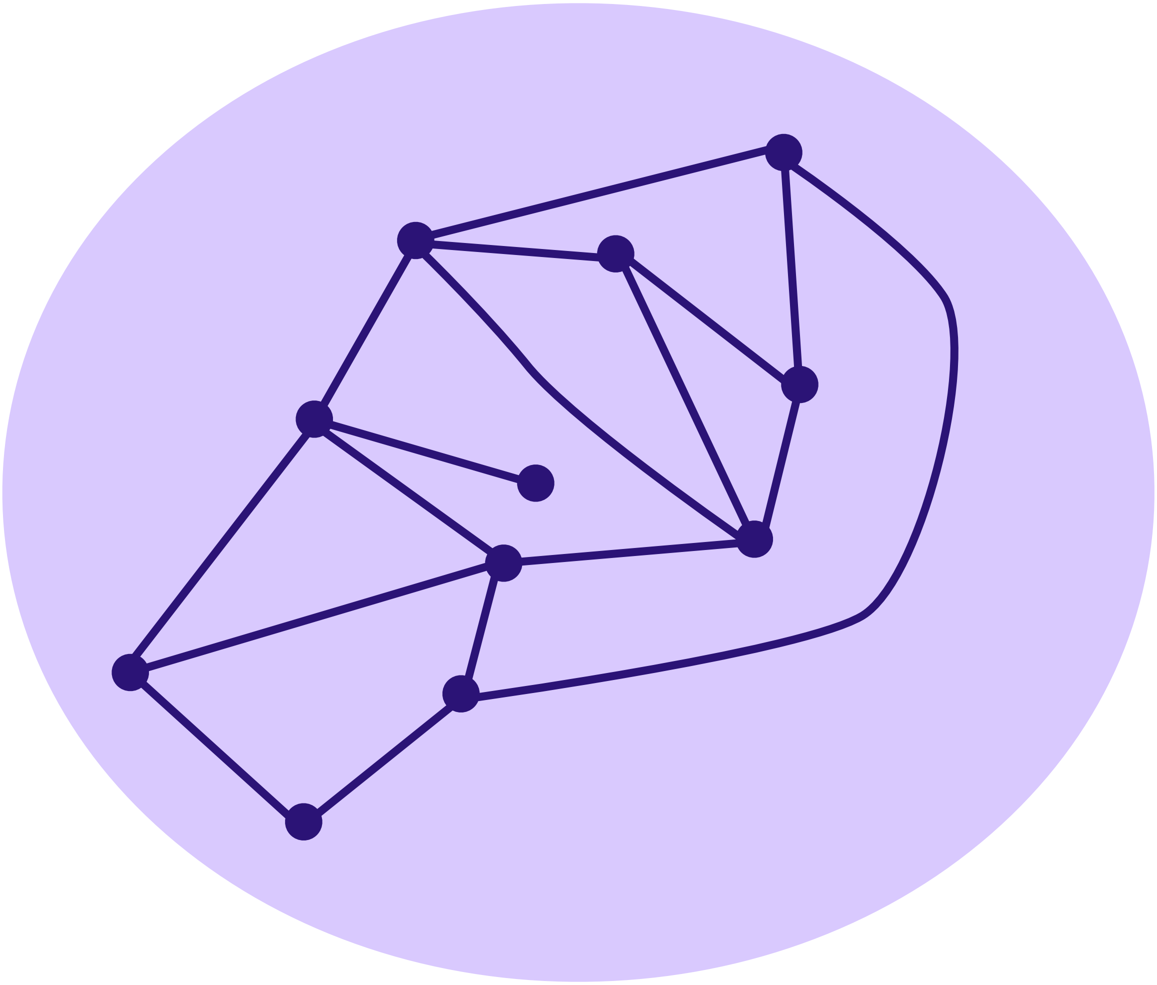
## Definition of a map

A map is a discrete surface obtained by gluing along edges faces with topology of a disk



weight
$t_k$
$(k \geq 1)$

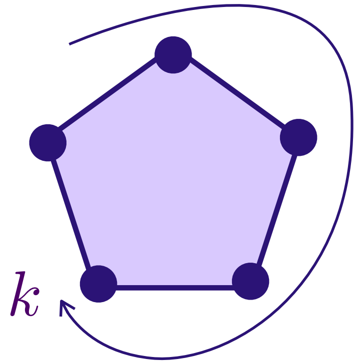
weight
per vertex
$t$



weight $= t^{11} t_3^3 t_4^2 t_5 t_6^2$

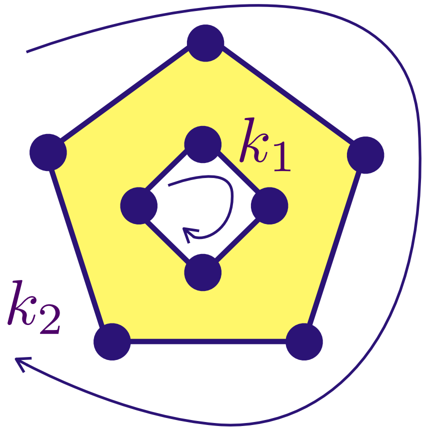
# Definition of a map with tubes

A map with tubes is a discrete surface obtained by gluing along edges faces with topology of a disk



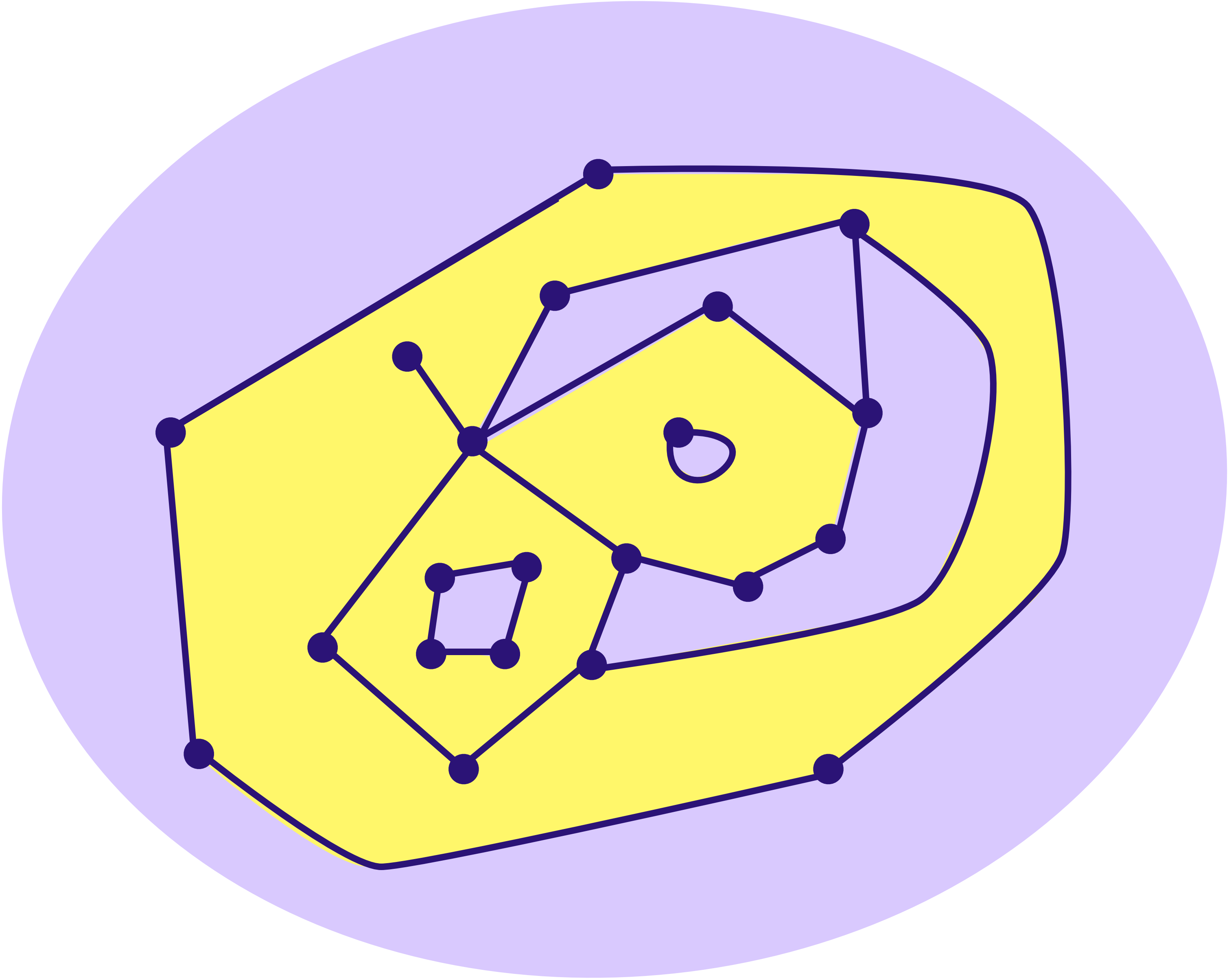
weight
$t_k$
$(k \geq 1)$

or with topology of a cylinder



weight
$\gamma t_{k_1,k_2}$
$(k_1 + k_2 \geq 1)$

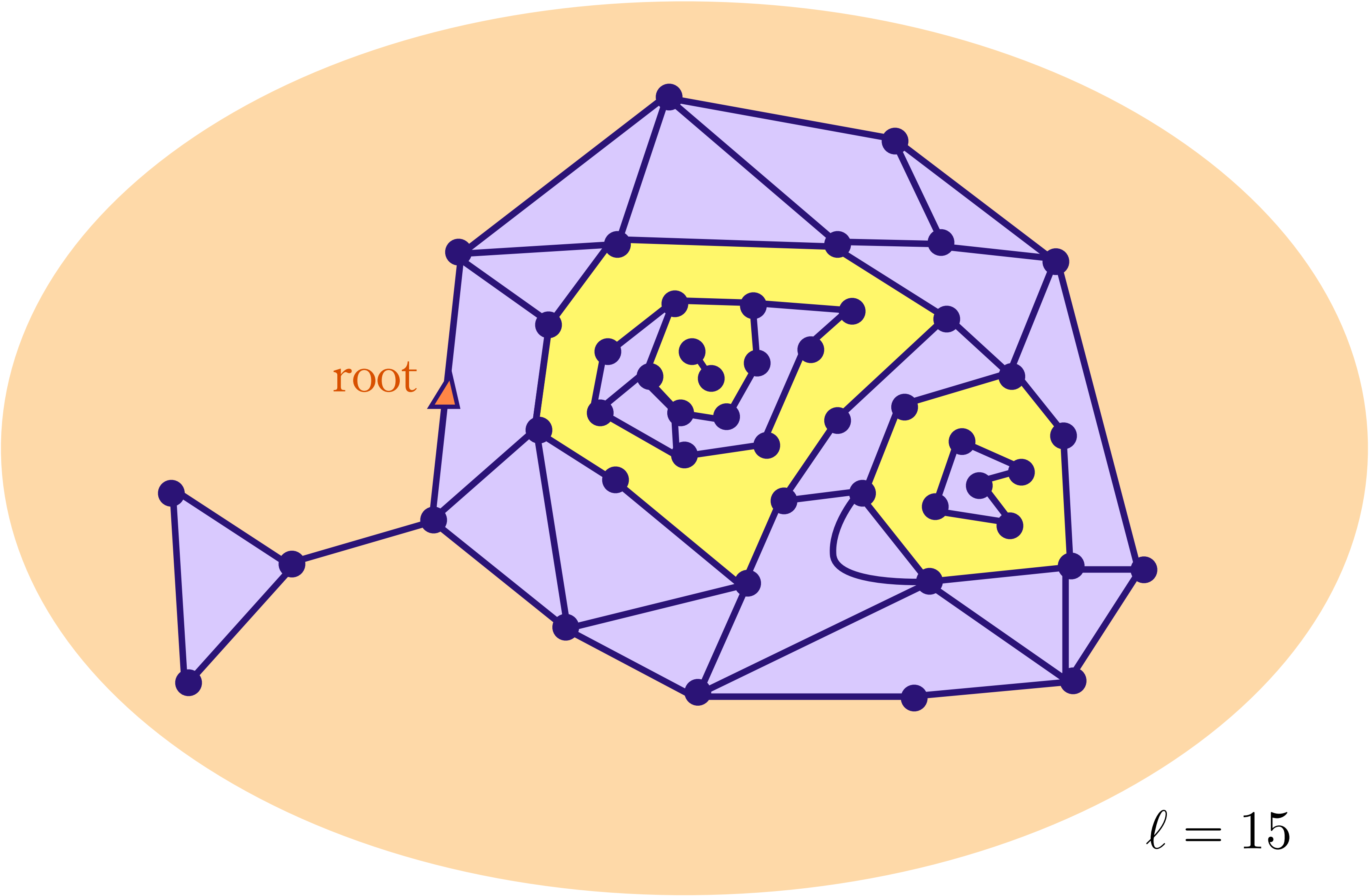
● weight $t$ per vertex



weight $= t^{22}\gamma^3 t_1 t_4^2 t_5 t_6 t_{4,8} t_{4,5} t_{6,1}$

Planar maps are those which can be embedded in a sphere.

$T_\ell[(t_k)_k;(t_{k_1,k_2})_{k_1,k_2}]$ = generating series of maps with 1 marked, rooted face of perimeter $\ell$



Planar

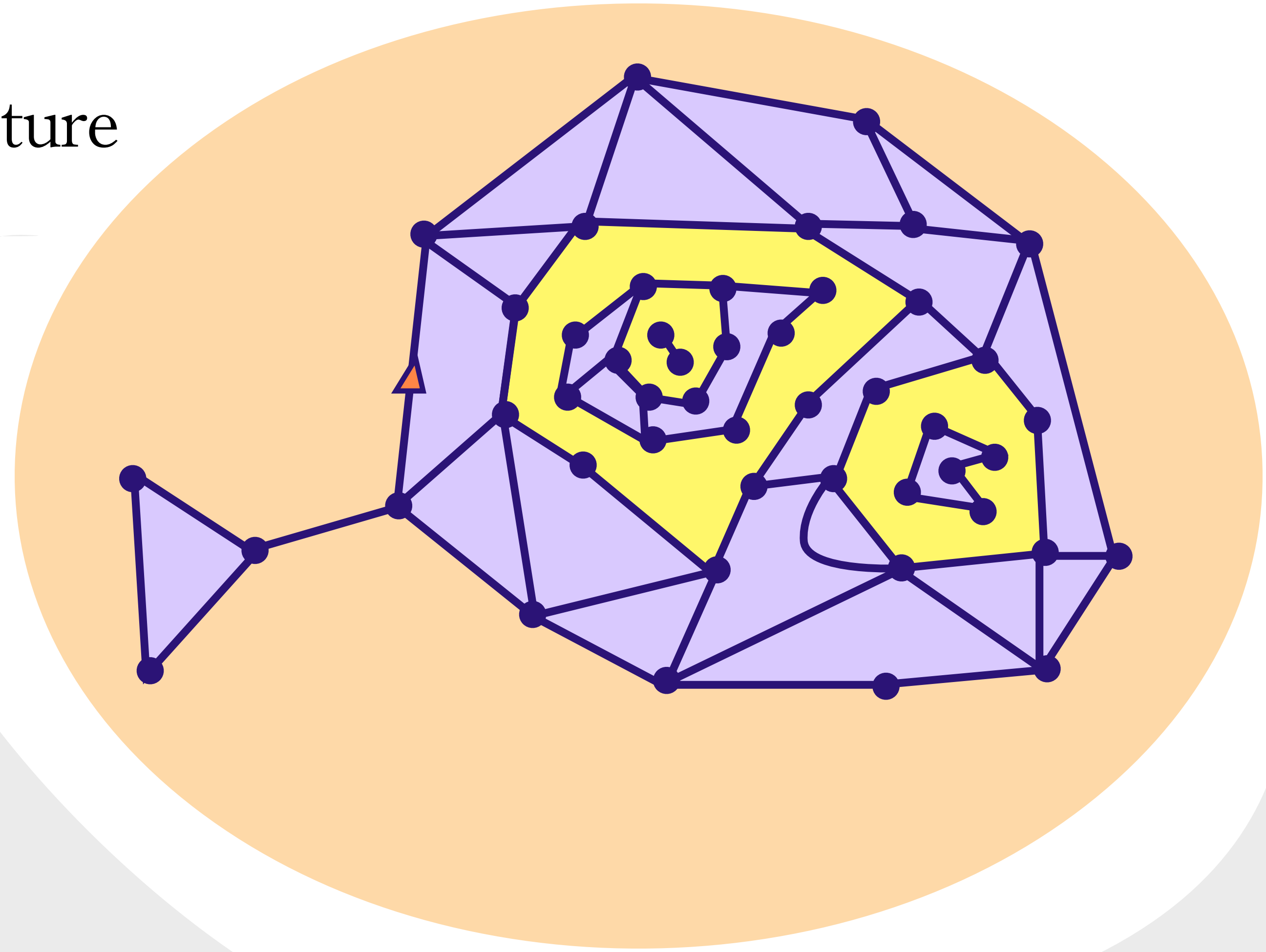
$\downarrow$

Notion of inside/outside a tube

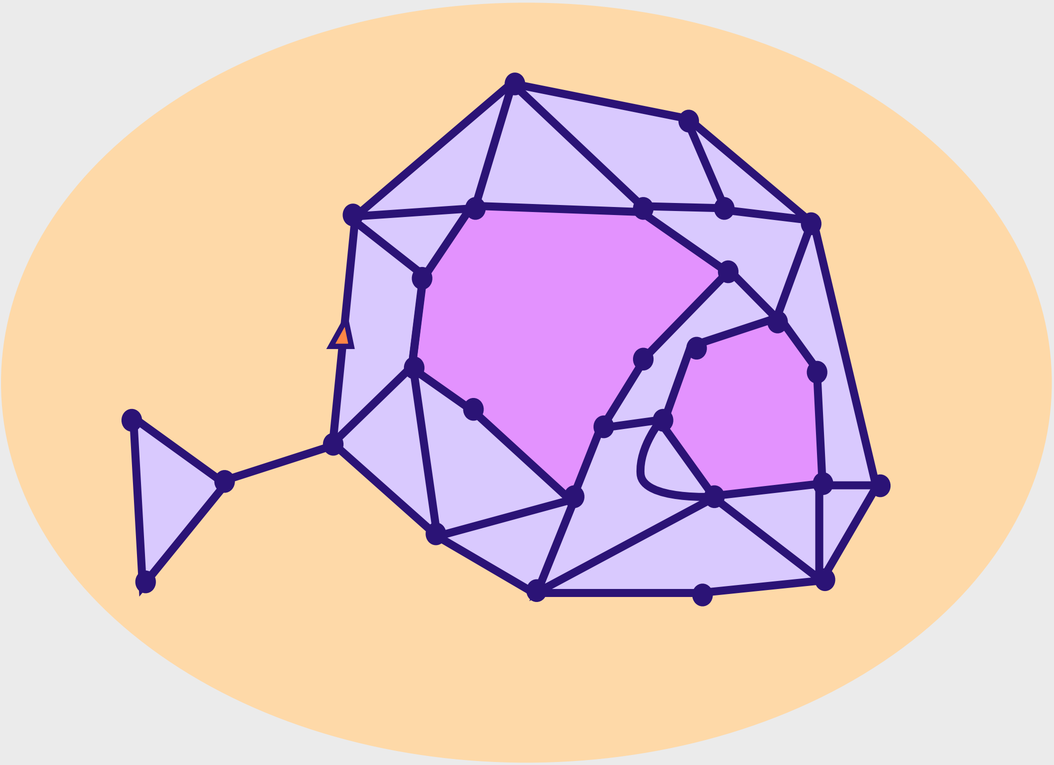
$\downarrow$

Planar maps with tubes have a nested structure

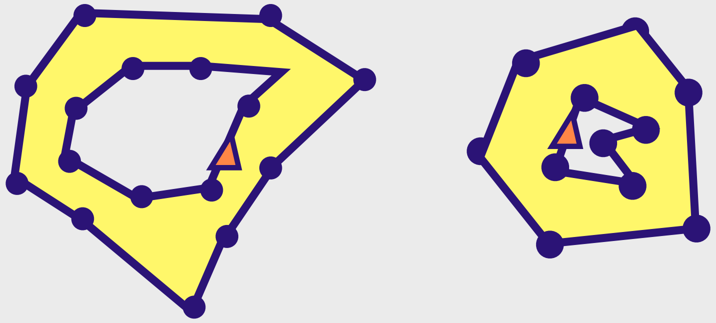
Planar maps with tubes have a nested structure



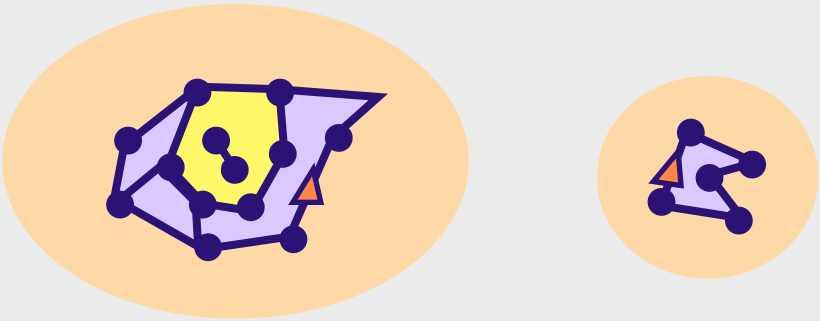
map with large faces (gasket)



with substitution of tubes (rooted inside)



stuffed with rooted maps with tubes

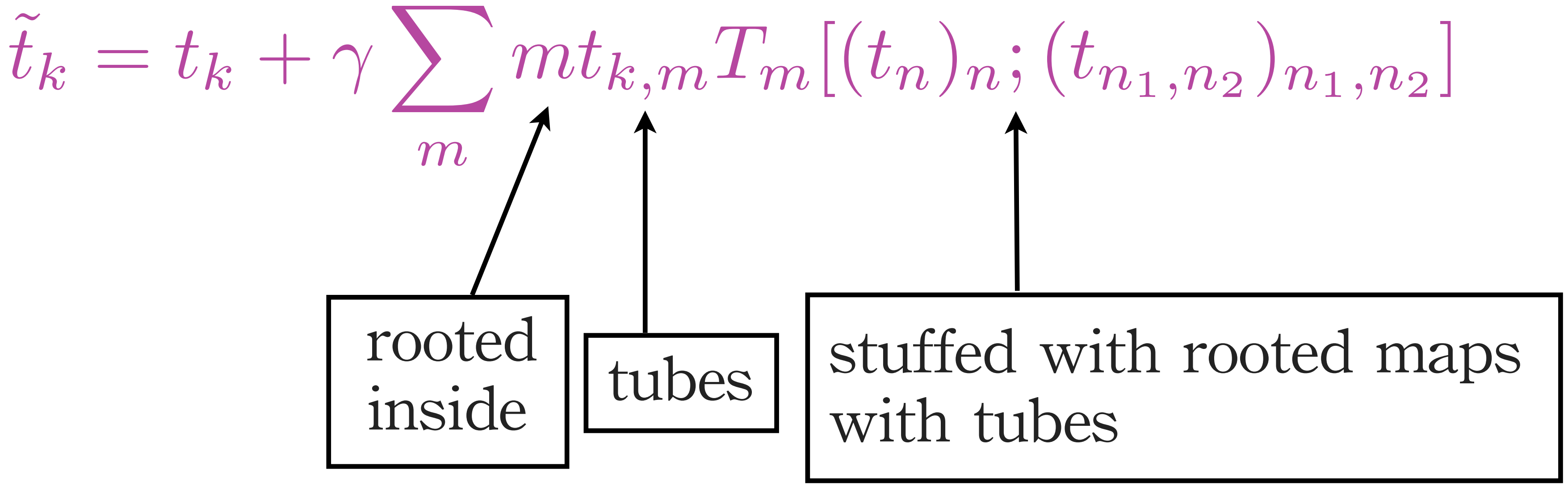


# Planar maps and substitution

Planar maps with tubes have a nested structure

$$T_\ell[(t_k)_k;(t_{k_1,k_2})_{k_1,k_2}] = T_\ell[(\tilde{t}_k)_k;0]$$

with substitution of



$$\tilde{t}_k = t_k + \gamma \sum_m m t_{k,m} T_m[(t_n)_n;(t_{n_1,n_2})_{n_1,n_2}]$$

rooted inside

tubes

stuffed with rooted maps with tubes

⟶ Counting planar maps with tubes is reduced to counting planar maps

# Admissible weights

## Definitions

$t, (t_k)_k$ non-negative is admissible when $\forall l \geq 1, \;\; t\partial_t T_\ell[(t_k)_k; 0] < +\infty$
(generating series of pointed rooted planar maps)

$t, (t_k)_k$ real-valued is admissible when $|t|, (|t_k|)_k$ is admissible

$t, (t_k)_k, \gamma, t_{k_1,k_2}$ is admissible when $t, (\tilde{t}_k)_k$ is admissible

# Planar 1-cut lemma

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$t, (t_k)_k\,, \gamma, t_{k_1,k_2}$ is admissible when $t, (\tilde{t}_k)_k$ is admissible

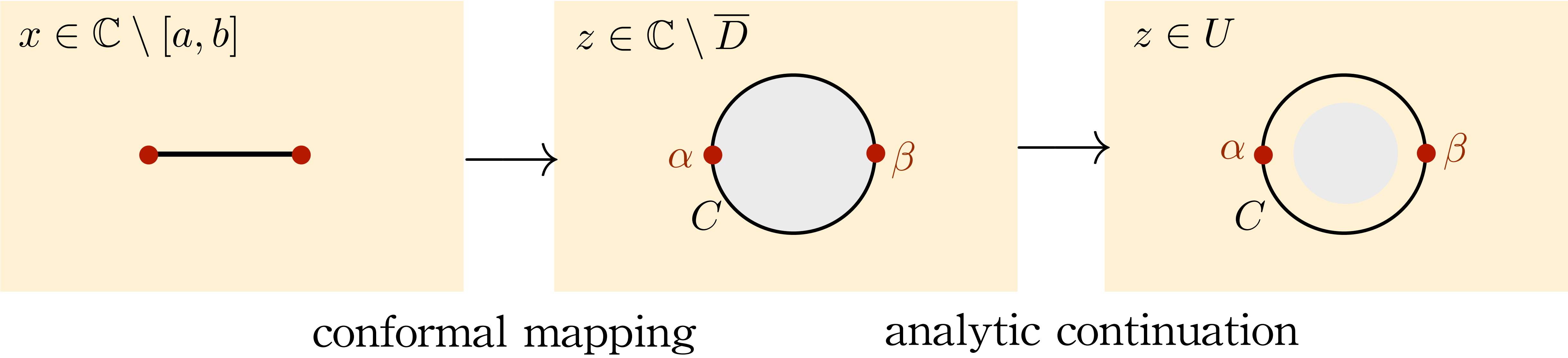
## Planar 1-cut lemma

Bousquet-Mélou (02), GB, Bouttier, Guitter (12)

If $t, (t_k)_k\,, \gamma, t_{k_1,k_2}$ is admissible, there exists $a, b \in \mathbb{R}$
so that $W_1^0(x) = \Big(\frac{t}{x} + \sum_{\ell \geq 1} \frac{T_\ell[(t_k)_k; (t_{k_1,k_2})_{k_1,k_2}]}{x^{\ell+1}}\Big)\mathrm{d}x$

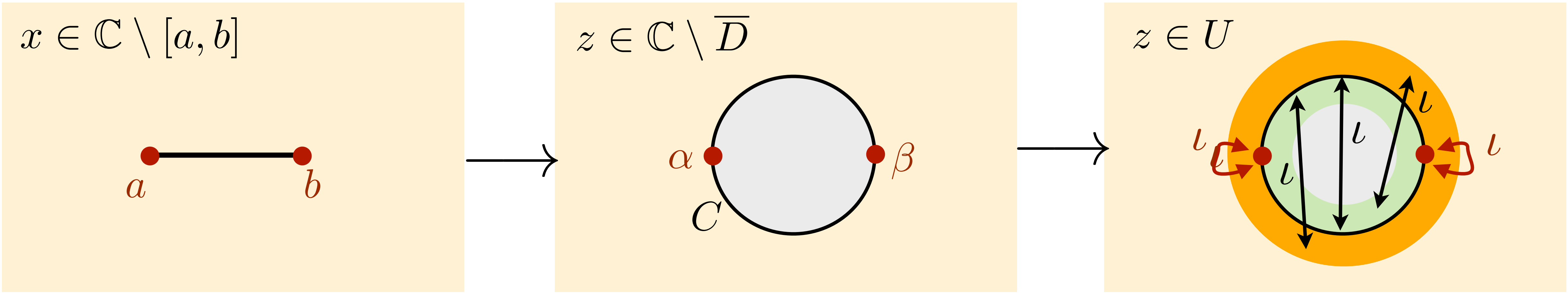
- ✷ is holomorphic in $\mathbb{C} \setminus [a, b]$
- ✷ has a discontinuity on $]a, b[$
- ✷ remains bounded

# Analytic continuation



**Fact** $\omega_1^0(z) = W_1^0(x(z))$ can be continued in a neighborhood $U$ of $C$

# Analytic continuation



conformal mapping analytic continuation

**Fact** $\omega_1^0(z) = W_1^0(x(z))$ can be continued in a neighborhood $U$ of $C$

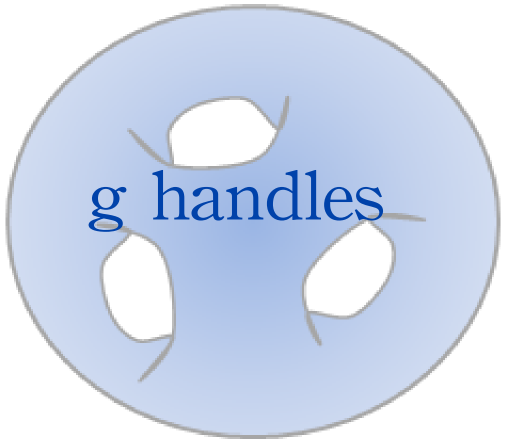
We have an involution $z \mapsto \iota(z)$ so that $x(z) = x(\iota(z))$ exchanging interior/exterior of $C$

$W_1^0(x)$ determines a Riemann surface, called spectral curve on which it can be seen as an analytic 1-form

# 1. Maps with(out) tubes

## Higher topologies

A map with(out) tubes has genus g if it is connected and can be embedded in



We define the generating series

$$W_n^g(x_1,\ldots,x_n) = \sum_{\ell_1,\ldots,\ell_n\geq 1}\Big[\prod_{i=1}^{n}\frac{\mathrm{d}x_i}{x_i^{\ell_i+1}}\Big]\times\left\{\begin{array}{l}\text{generating series of genus g maps}\\ \text{with rooted marked faces}\\ \text{of perimeters } \ell_1,\ldots,\ell_n\end{array}\right\}$$

and we would like to compute them ...

We cannot speak of nesting for $g > 0$ !

Rather use Tutte bijective decomposition of maps to establish functional relations

+ a good deal of complex analysis ...

# 1. Maps with(out) tubes

$$W_n^g(x_1,\ldots,x_n) = \sum_{\ell_1,\ldots,\ell_n\geq 1} \Big[\prod_{i=1}^{n} \frac{\mathrm{d}x_i}{x_i^{\ell_i+1}}\Big] \times \left\{\begin{array}{l}\text{generating series of genus g maps}\\ \text{with rooted marked faces}\\ \text{of perimeters } \ell_1,\ldots,\ell_n\end{array}\right\}$$

**1-cut lemma**

If $t,(t_k)_k,\gamma,t_{k_1,k_2}$ is admissible, then $W_n^g(x_1,\ldots,x_n)$

- ✷ is holomorphic in $\mathbb{C}\setminus[a,b]$
- ✷ has a discontinuity when $x_i\in[a,b]$
- ✷ can be analytically continued to $\omega_n^g(z_1,\ldots,z_n)$ on the same Riemann surface

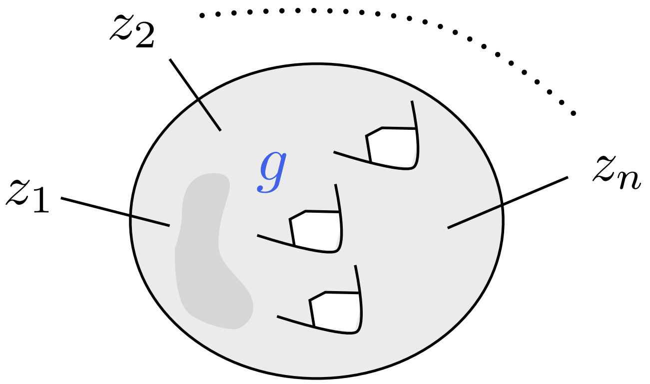
It can be computed by a recursion on 2g - 2 + n
which takes a universal form

for maps    Eynard (06)

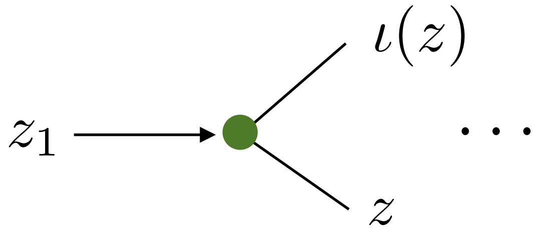
for maps with tubes  GB, Eynard, Orantin (13)

It can be computed by a recursion of topological nature.

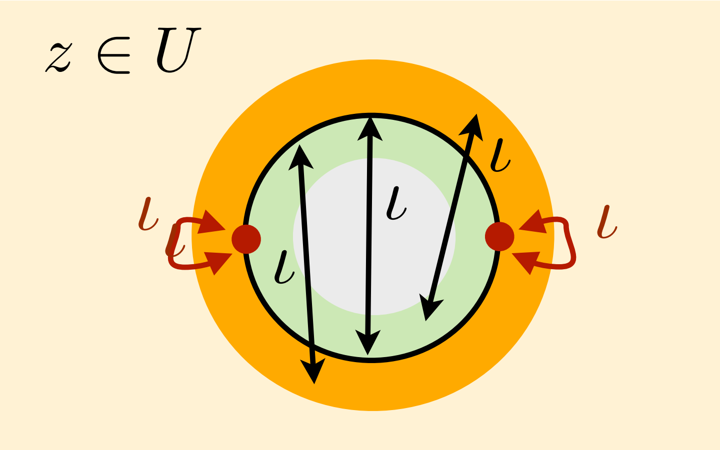
$$\omega_n^g(z_1,\ldots,z_n) =$$

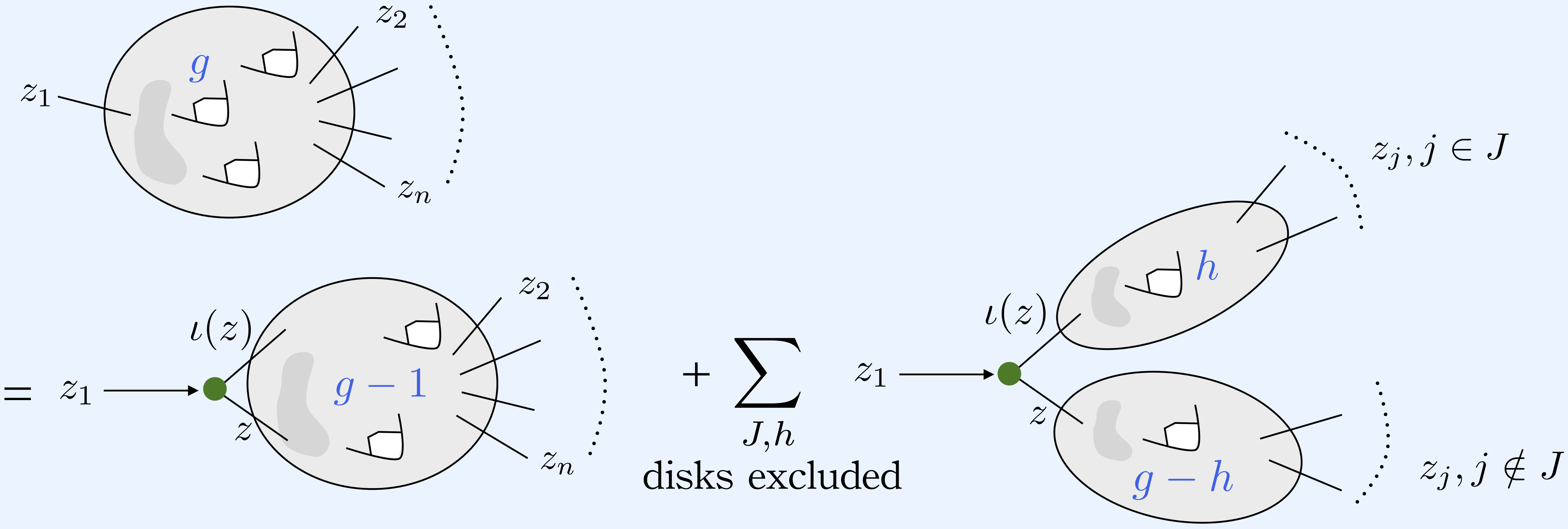


$$\operatorname*{Res}_{z\to\alpha,\beta} \frac{-\frac{1}{2}\int_{\iota(z)}^{z}\omega_2^0(\cdot,z_2)}{\omega_1^0(z)-\omega_1^0(\iota(z))}\cdots = $$



the involution $\iota$

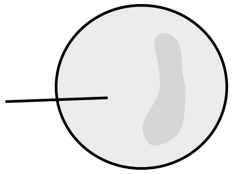
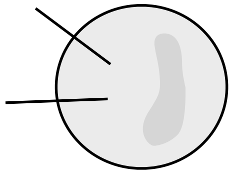




disks excluded

# Topological recursion

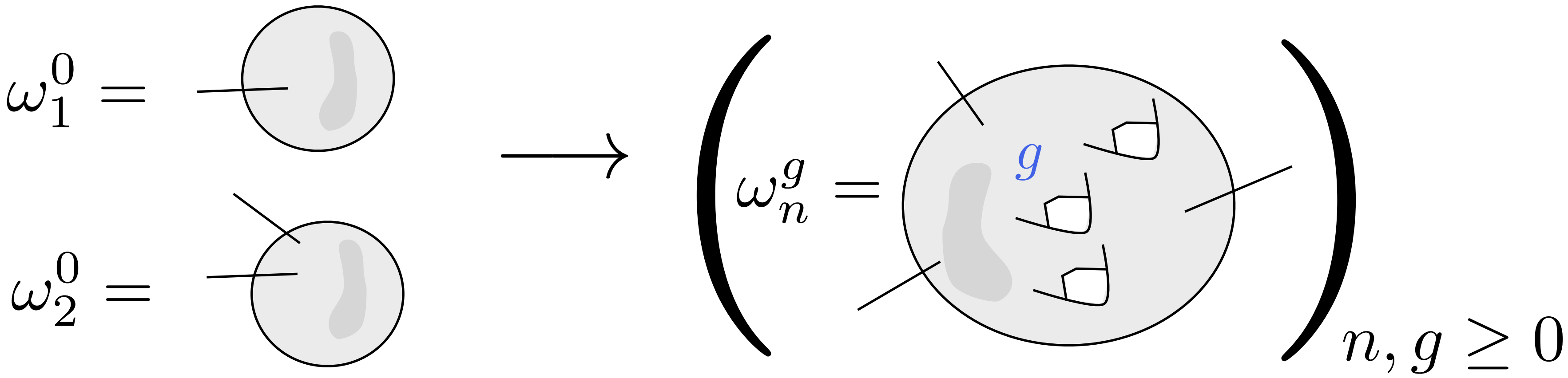
$\omega_n^g$ can be computed by a universal recursion of topological nature

if one knows $\omega_1^0 =$  and $\omega_2^0 =$ 

but the combinatorial interpretation of the recursion is not known ...

$\omega_n^g(\iota(z), z_2, \ldots, z_n)$ ??

**Definition** The topological recursion is the algorithm Eynard, Orantin (07)



$$\omega_1^0 = \qquad \omega_2^0 = \qquad \longrightarrow \left( \omega_n^g = \right)_{n, g \geq 0}$$

initial data

output

# 2 … and formal matrix models

Relation to maps

Multidimensional integrals …

… and their asymptotics

# 2. Formal matrix models

Generating series of maps can be represented by formal matrix integrals

Brézin, Itzykson, Parisi, Zuber (78)

if we introduce

the generating series of faces

$$D(x) = \frac{N}{t} \sum_k \frac{t_k}{k} x^k$$

the gaussian measure on $N \times N$ hermitian matrices

$$\mathrm{d}\mu(M) = \mathrm{d}M \, e^{-N \mathrm{Tr} M^2/2t}$$

then, the formal series in $t, t_k$ has a well-defined decomposition

$$\frac{\mu\Big[e^{\mathrm{Tr} D(M)} \prod_{i=1}^n \mathrm{Tr} \, \frac{\mathrm{d}x_i}{x_i - M}\Big]_c}{\mu\Big[e^{\mathrm{Tr}\, D(M)}\Big]} = \sum_{g \geq 0} N^{2-2g-n} \, W_n^g(x_1, \ldots, x_n)$$

generating series of maps
of genus g with n rooted marked faces

# 2. Formal matrix models … and maps with tubes

Generating series of maps with tubes have a similar representation

reformulation of Gaudin, Mehta, Kostov (90s)

if we introduce the generating series of faces

with topology of a disk $$D(x) = \frac{N}{t}\sum_{k}\frac{t_k}{k}\,x^k$$

with topology of a cylinder $$C(x,y) = \frac{\gamma}{t}\sum_{k_1,k_2}\frac{t_{k_1,k_2}}{2k_1k_2}\,x^{k_1}y^{k_2}$$

then, the formal series in $t, t_k, \gamma, t_{k_1,k_2}$ has a well-defined decomposition

$$\frac{\mu\Big[e^{\mathrm{Tr}D(M)+\mathrm{Tr}C(M\otimes\mathbf{1}_N,\mathbf{1}_N\otimes M)}\,\prod_{i=1}^{n}\mathrm{Tr}\,\frac{\mathrm{d}x_i}{x_i-M}\Big]_c}{\mu\Big[e^{\mathrm{Tr}\,D(M)+\mathrm{Tr}C(M\otimes\mathbf{1}_N,\mathbf{1}_N\otimes M)}\Big]} = \sum_{g\geq 0} N^{2-2g-n}\,W_n^g(x_1,\ldots,x_n)$$

generating series of maps with tubes
of genus g with n rooted marked faces

# 2. Formal matrix models

## Multidimensional integrals

U(N) invariant measures :

the integrals over M reduces to an integral over its eigenvalues

e.g. the partition function

$$\mu\left[e^{\mathrm{Tr}\,D(M)+\mathrm{Tr}C(M\otimes\mathbf{1}_N,\mathbf{1}_N\otimes M)}\right]$$

$$=\frac{\mathrm{Vol}(U(N))}{N!(2\pi)^N}\int_{\mathbb{R}^N}\prod_{i=1}^{N}\mathrm{d}\lambda_i e^{-\frac{N\lambda_i^2}{2t}+D(\lambda_i)+C(\lambda_i,\lambda_i)}\prod_{1\leq i<j\leq N}(\lambda_i-\lambda_j)^2\,e^{2C(\lambda_i,\lambda_j)}$$

arbitrary two-point interaction
vanishing like a square
at short distance

as presented for formal integrals

A summary on the large N expansion of $\int \prod_{i=1}^{N} \mathrm{d}\lambda_i e^{-NV(\lambda_i)} \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)^2 K(\lambda_i, \lambda_j)$

K non-vanishing

formal integral ($\longrightarrow$ combinatorics, 1/N expansion, 1-cut lemma)

K = 1 Ambjørn, Makeenko, Chekhov, Kristjansen (90s), Eynard (04)

K $\neq$ 1 GB, Eynard, Orantin (13)

cv integral + 1-cut + assumptions $\longrightarrow$ 1/N expansion

K = 1 Albeverio, Pastur, Shcherbina (00), Ercolani, McLaughlin (04)
GB, Guionnet (11)

K $\neq$ 1 GB, Guionnet, Kozlowski (in progress)

Topological recursion

cv integral + several cuts + assumptions $\longrightarrow$ ~~1/N expansion~~

K = 1 heuristics : Bonnet, David, Eynard (00), Eynard (07)
proof : GB, Guionnet (13)

K $\neq$ 1 GB, Guionnet, Kozlowski (in progress)

Topological recursion with nodes

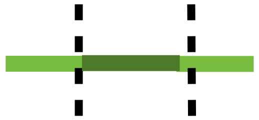
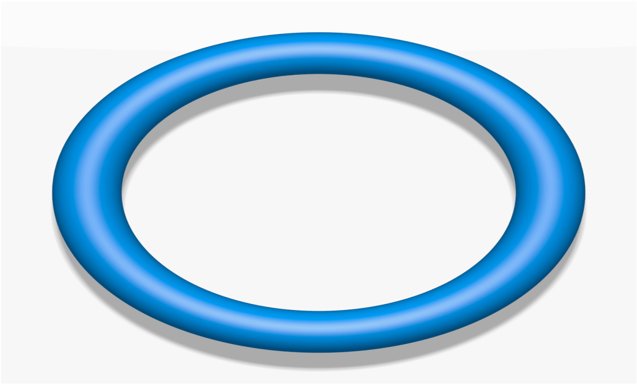
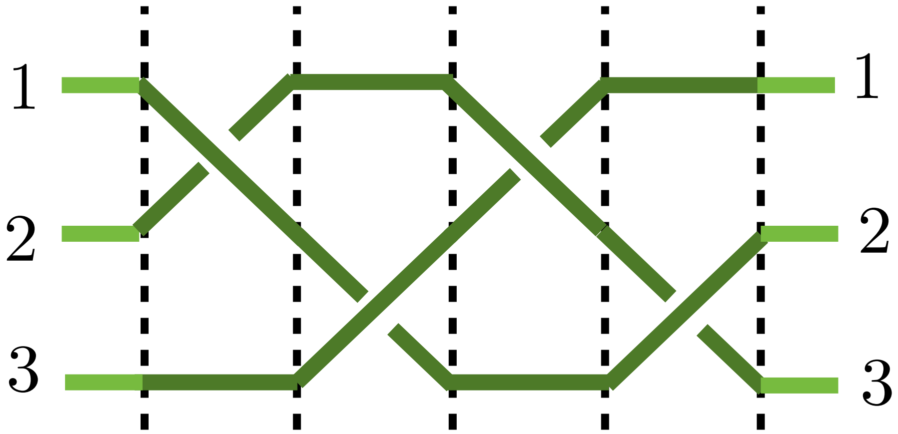
(see later ...)

> Imite le moins possible les hommes dans leur
> énigmatique maladie de faire des nœuds.
>
> René Char, Rougeurs des matinaux

# 3 Introduction to knots

- Definition, classification
- Knot invariants
- Asymptotics and why should we care ?

A knot is an isotopy class of proper embedding of a circle in $\mathbb{S}^3$

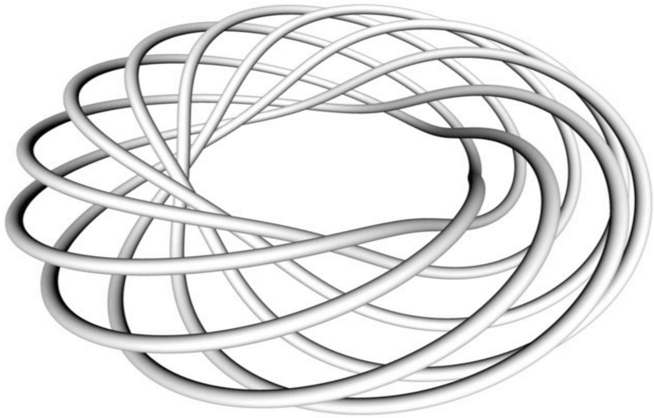
| Word in the braid group ↔ 2d projection of a knot | Tubular neighborhood of a knot | Example |
|---|---|---|
|  |  | unknot |
|  |  | figure-eight knot |

# Classification of knots

Knots are complicated (as much as arithmetic in $\mathbb{Z}$ is ...)

There exist infinitely many prime knots, which fall in 3 families

- (P,Q) torus knots



- hyperbolic knots $\mathbb{S}_3 \setminus K$ admits a complete hyperbolic metric

- satellite knots
  (uncharted territory)

## Knot invariants

Knots are complicated (as much as arithmetic in $\mathbb{Z}$ is ...)

Hard algorithmic problems :

- ✷ unknotting number
- ✷ distinguishing two knots

$\longrightarrow$ construction of (computable) knot invariants to give partial answers

For any compact Lie group G , irreducible representation R ,
one can construct an invariant using representations of quantum groups

$$K \longmapsto W_K(G,R;q) \in \mathbb{Z}[q,q^{-1}]$$

knot colored HOMFLY polynomial

behaving nicely under geometric operations (gluings, cabling, ...)

## Knot invariants

For any compact Lie group $G$, irreducible representation $R$,
one can construct an invariant using representations of quantum groups

$$K \longmapsto W_K(G,R;q) \in \mathbb{Z}[q,q^{-1}]$$

knot colored HOMFLY polynomial

behaving nicely under geometric operations (gluings, cabling, ...)

$G = SU(2)$ $R = \square$ Jones polynomial

$R = \underbrace{\square\square\cdots\square}_{(m\text{ - }1)\text{ boxes}}$ m-th colored Jones polynomial $= J_{K,m}(q)$

$$J_m^{\text{unknot}}(q) = \frac{q^m - q^{-m}}{q - q^{-1}}$$

$$J_m^{8-\text{knot}}(q) = \frac{q^m - q^{-m}}{q - q^{-1}} \left( \sum_{k=0}^{m-1} (q^2)_k (1/q^2)_k \right)$$

but in general no closed formulas ...

# 3. Introduction to knots — Asymptotics of knot invariants

Enumeration of knots by matrix model techniques
Zinn-Justin, Zuber

has nothing to do with the topic of this talk

For a given knot $K$
we would like to compute the asymptotic expansion of $W_K(G,R;q)$ when

- $G = SU(N)$ $\qquad N \to \infty \quad q \to 1$

  $R =$ fixed Young tableau $\qquad t = N \ln(q)$ fixed

  (theory of LMO invariants)

- $G = SU(2)$ $\qquad m \to \infty \quad q \to 1$

  $R = \square\square\cdots\square$ $\qquad u = m \ln(q)$ fixed

  (m - 1) boxes

# Asymptotics of knot invariants

For a given knot $K$

we would like to compute the asymptotic expansion of $W_K(G,R;q)$ when

✷ $G = SU(N)$ $\qquad$ $N \to \infty \quad q \to 1$

$R =$ fixed Young tableau $\qquad$ $t = N \ln(q)$ fixed

(theory of LMO invariants)

4. Theorem : for torus knots, by the **topological recursion**

✷ $G = SU(2)$ $\qquad$ $m \to \infty \quad q \to 1$

$R = \square\square\cdots\square$ $\qquad$ $u = m \ln(q)$ fixed

(m - 1) boxes

5. Conjecture : for hyperbolic knots, by the **topological recursion with nodes**

# 3. Introduction to knots

## Why should we care ?

The interest about asymptotics of knot invariants started from

✷ $G = SU(2)$ $\quad\quad$ $m \to \infty \quad q \to 1$

$R = \underbrace{\square\square\cdots\square}_{(m\ -\ 1)\text{ boxes}}$ $\quad\quad$ $u = m\ln(q)$ fixed

**Volume conjecture** Kashaev (98), H. Murakami (00)

If K is a hyperbolic knot, $\lim_{m\to\infty}\frac{2\pi}{m}\ln\left|J_{K,m}(q=e^{2i\pi/m})\right| = \mathrm{Volume}(\mathbb{S}_3\setminus K)$

Algebraic construction ⟶ Geometric information

# 3. Introduction to knots

## Why should we care ?

The interest about asymptotics of knot invariants started from

✷ $G = SU(2)$ $\qquad$ $m \to \infty \quad q \to 1$

$R = \underbrace{\square\square\cdots\square}_{(m\text{ - }1)\text{ boxes}}$ $\qquad$ $u = m\ln(q)$ fixed

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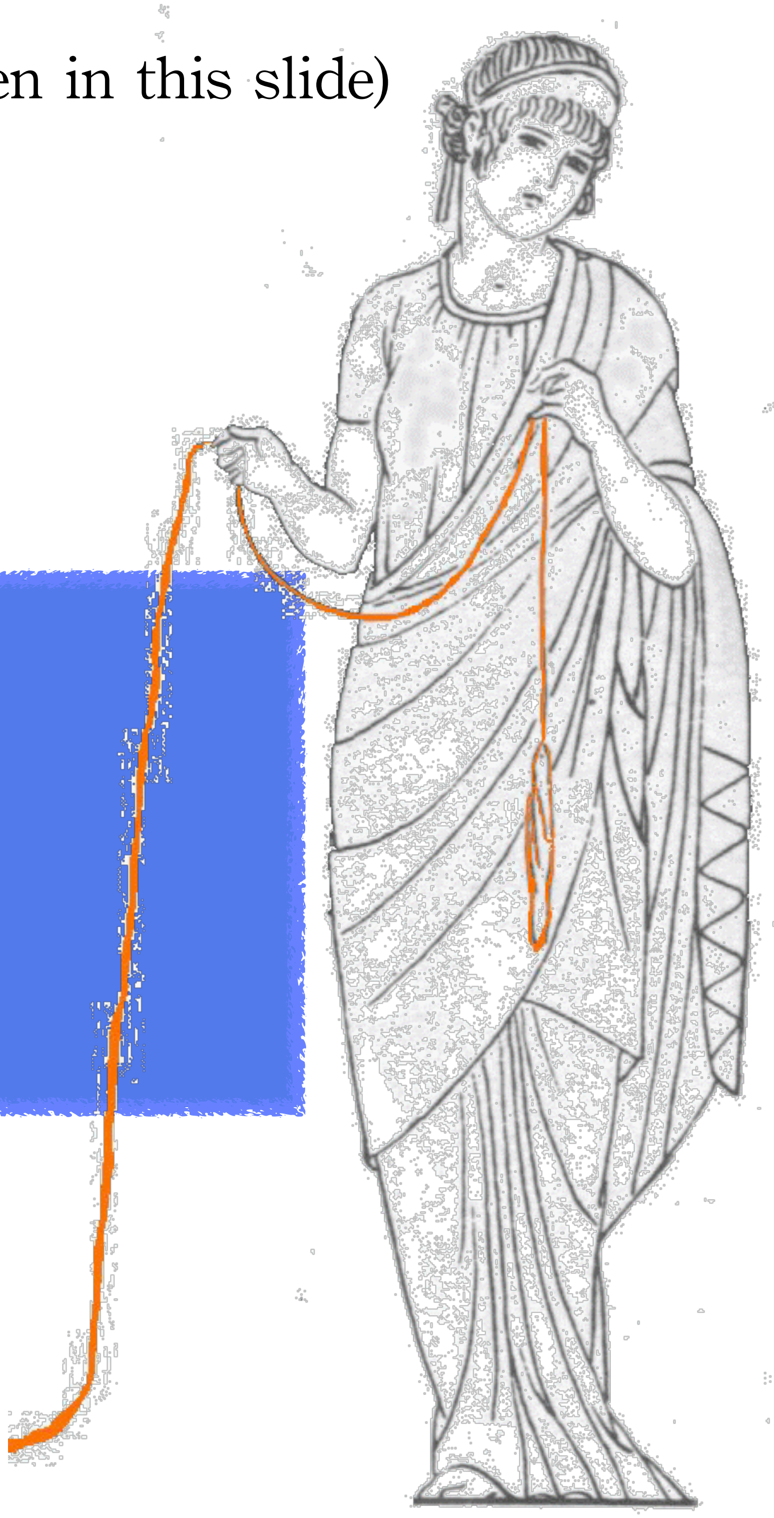
**Other conjectures** other values of $u$ and asymptotic expansion Gukov (04)
arithmeticity, modularity, ... Zagier et al. (09)

Computation and unified understanding of those properties ?
Relation to other fields (counting surfaces ...) ?

(There is a pun hidden in this slide)

# 4 Torus knots and $W(G_{,R})$



Definition, classification

Knot invariants

Asymptotics and why should we care ?

# 4. Torus knots and $W(G,R)$

$W_K(G,R;q)$

can be computed as an observable in Chern-Simons theory in $\mathbb{S}_3$ Witten (89)

CS theory = quantum field theory with measure

$$\mathrm{d}\mu_{\mathrm{CS}}[\mathcal{A}] = D\mathcal{A}\,\exp\left\{-\frac{1}{\ln q}\left(\mathcal{A}\wedge \mathrm{d}\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right)\right\}$$

$\mathcal{A}$ = section of a $G$ principal bundle over $\mathbb{S}_3$

**Fact** If $C$ is a loop in $\mathbb{S}_3$

$\mu\big[\,\mathrm{Tr}_R(\text{holonomy of } \mathcal{A} \text{ along } C\,)\big]$ is a topological invariant

and coincides with $W_K(G,R;q)$

# The case of torus knots

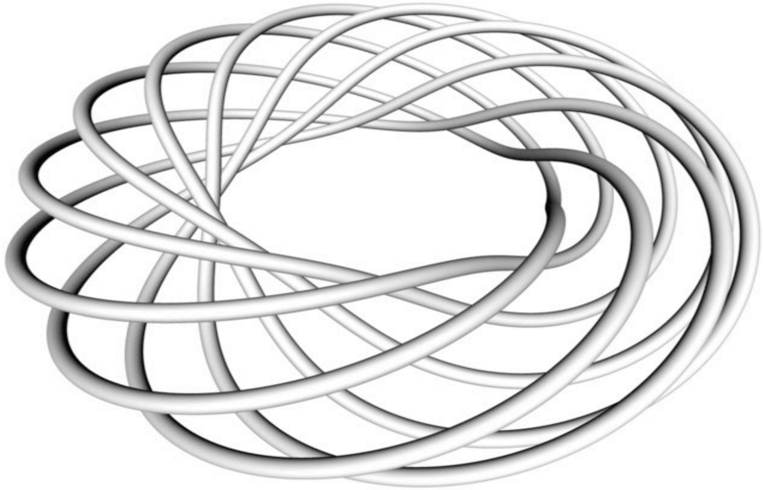
For torus knots, the path integral reduces to a finite dimensional integral !

(exact saddle point) Rozansky (98), Mariño (02), Beasley, Witten (07), Kállen (09)

$G = SU(N)$

$t = N \ln(q)$

$K =$



$$\mathrm{d}\mu_N^{P,Q} = \prod_{i=1}^{N} \mathrm{d}\lambda_i \, e^{-N\lambda_i^2/2PQt} \prod_{1\leq i<j\leq N} (\lambda_i - \lambda_j)^2 K(\lambda_i, \lambda_j)$$

with $K(x,y) = \dfrac{\sinh\left(\frac{x-y}{2P}\right)}{\left(\frac{x-y}{2P}\right)} \dfrac{\sinh\left(\frac{x-y}{2Q}\right)}{\left(\frac{x-y}{2Q}\right)}$

and the knot invariants are $W(SU(N),R;q) = \dfrac{\mu_N^{P,Q}\left[\mathrm{Schur}_R(e^{\lambda_1},\ldots,e^{\lambda_N})\right]}{\mu_N^{P,Q}[1]}$

# The case of torus knots

$$\mathrm{d}\mu_N^{P,Q} = \prod_{i=1}^{N} \mathrm{d}\lambda_i \, e^{-N\lambda_i^2/2PQt} \prod_{1\leq i<j\leq N} (\lambda_i - \lambda_j)^2 K(\lambda_i, \lambda_j)$$

with $K(x,y) = \dfrac{\sinh\left(\frac{x-y}{2P}\right)}{\left(\frac{x-y}{2P}\right)} \, \dfrac{\sinh\left(\frac{x-y}{2Q}\right)}{\left(\frac{x-y}{2Q}\right)}$

and the knot invariants are $W(SU(N),R;q) = \dfrac{\mu_N^{P,Q}\left[\mathrm{Schur}_R(e^{\lambda_1},\ldots,e^{\lambda_N})\right]}{\mu_N^{P,Q}[1]}$

It is convenient to use another basis of symmetric functions

$\mathrm{Schur}_R \longleftrightarrow$ power sums

and form a generating series of expectation values of power sums

$$W_n(x_1,\ldots,x_n) = \frac{\mu_N^{P,Q}\left[\prod_{i=1}^n \mathrm{Tr}\, \frac{\mathrm{d}x_i}{x_i - e^{M/PQ}}\right]_c}{\mu_N^{P,Q}[1]} \qquad M = \mathrm{diag}(e^{\lambda_1},\ldots,e^{\lambda_N})$$

# 4. Torus knots and W(G,R)

$$\mathrm{d}\mu_N^{P,Q} = \prod_{i=1}^{N} \mathrm{d}\lambda_i \, e^{-N\lambda_i^2/2PQt} \prod_{1\leq i<j\leq N} (\lambda_i - \lambda_j)^2 K(\lambda_i, \lambda_j)$$

with $K(x,y) = \frac{\sinh\left(\frac{x-y}{2P}\right)}{\left(\frac{x-y}{2P}\right)} \frac{\sinh\left(\frac{x-y}{2Q}\right)}{\left(\frac{x-y}{2Q}\right)}$

We would like to compute $W_n(x_1, \ldots, x_n) = \frac{\mu_N^{P,Q}\left[\prod_{i=1}^{n} \mathrm{Tr}\, \frac{\mathrm{d}x_i}{x_i - e^{M/PQ}}\right]_c}{\mu_N^{P,Q}[1]}$

**Theorem**

prediction, check : Brini, Eynard, Mariño (11)
proof : GB, Eynard, Orantin (13)

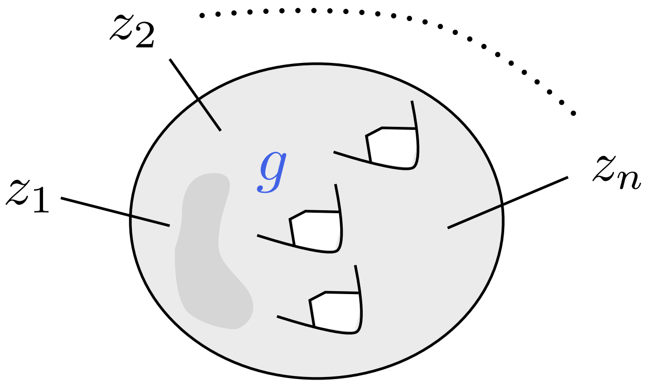
Assume $t = N \ln q > 0$ fixed

- There is a large N expansion $W_n = \sum_{g\geq 0} N^{2-2g-n}\, W_n^g$
- There is an explicit formula for $W_1^0$ and $W_2^0$
- $W_n^g$ have 1-cut $[a,b]$, and are computed by the topological recursion

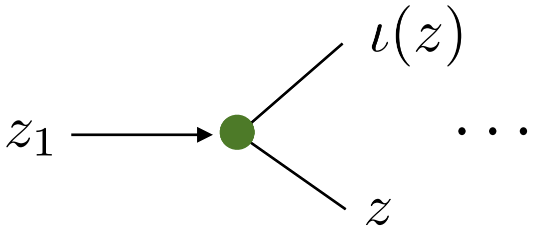
# Reminder: topological recursion

It can be computed by a recursion of topological nature.

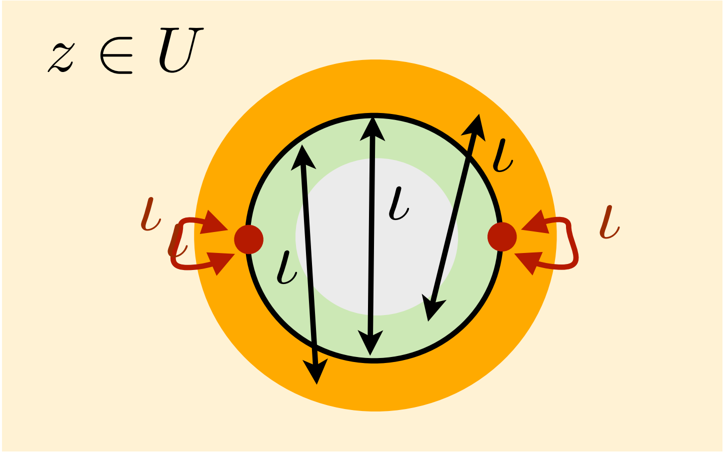
$$\omega_n^g(z_1,\ldots,z_n) =$$

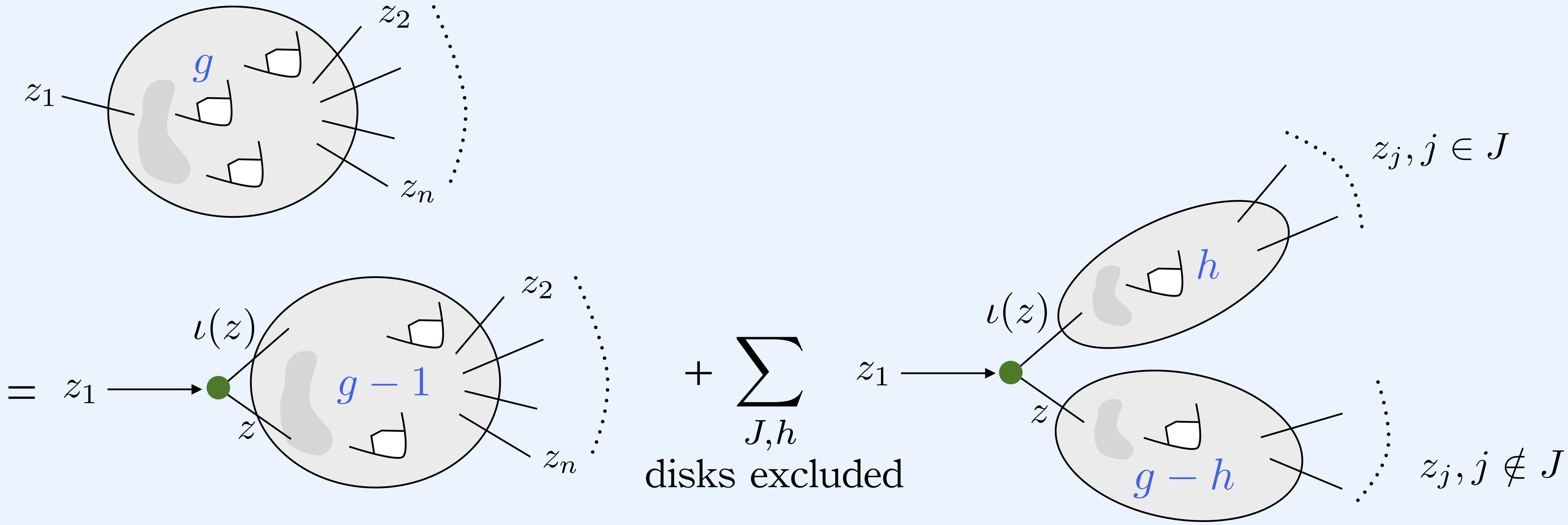


$$\mathop{\mathrm{Res}}_{z\to\alpha,\beta} \frac{-\frac{1}{2}\int_{\iota(z)}^{z} \omega_2^0(\cdot, z_2)}{\omega_1^0(z) - \omega_1^0(\iota(z))} \cdots =$$



the involution $\iota$





$+ \sum_{J,h}$
disks excluded

## Torus knots: conclusion

$$\mathrm{d}\mu_N^{P,Q} = \prod_{i=1}^{N} \mathrm{d}\lambda_i\, e^{-N\lambda_i^2/2PQt} \prod_{1\leq i<j\leq N} (\lambda_i - \lambda_j)^2 K(\lambda_i, \lambda_j)$$

with $K(x,y) = \dfrac{\sinh\left(\frac{x-y}{2P}\right)}{\left(\frac{x-y}{2P}\right)} \dfrac{\sinh\left(\frac{x-y}{2Q}\right)}{\left(\frac{x-y}{2Q}\right)}$

We would like to compute $W_n(x_1,\ldots,x_n) = \dfrac{\mu_N^{P,Q}\left[\prod_{i=1}^{n} \mathrm{Tr}\, \frac{\mathrm{d}x_i}{x_i - e^{M/PQ}}\right]_c}{\mu_N^{P,Q}[1]}$

✸ This model is a special case of
the matrix model enumerating maps with tubes

any explanation ??

✸ Cannot be generalized yet to hyperbolic knots ...
(no finite-dimensional reduction of CS theory)

# 5 Hyperbolic knots and $W(G, \mathbb{R})$

Definition, classification

Knot invariants

Asymptotics and why should we care ?

# 5. Hyperbolic knots and W(G,R)

## Generalized vol. conjecture

The interest about asymptotics of knot invariants started from

✷ $G = SU(2)$ $\qquad$ $m \to \infty \quad q \to 1$

$R = \square\square\cdots\square$ $\qquad$ $u = m\ln(q)$ fixed

(m - 1) boxes

**Volume conjecture** Kashaev (98), H. Murakami (00)

If K is a hyperbolic knot, $\displaystyle\lim_{m\to\infty}\frac{2\pi}{m}\ln\left|J_{K,m}(q=e^{2i\pi/m})\right| = \text{Volume}(\mathbb{S}_3\setminus K)$

## Generalized vol. conjecture

The interest about asymptotics of knot invariants started from

- $G = SU(2)$ $\quad\quad m \to \infty \quad q \to 1$

  $R = \underbrace{\square\square\cdots\square}_{(m\,-\,1)\text{ boxes}}$ $\quad\quad u = m\ln(q)$ fixed

**Generalized ...** Gukov (04)

If K is a hyperbolic knot,

$$J_{K,m}(q) = (\ln q)^{\Delta/2} \exp\left( \sum_{k \geq -1} (\ln q)^k \, S_k(u) + o(\ln q)^{\infty} \right)$$

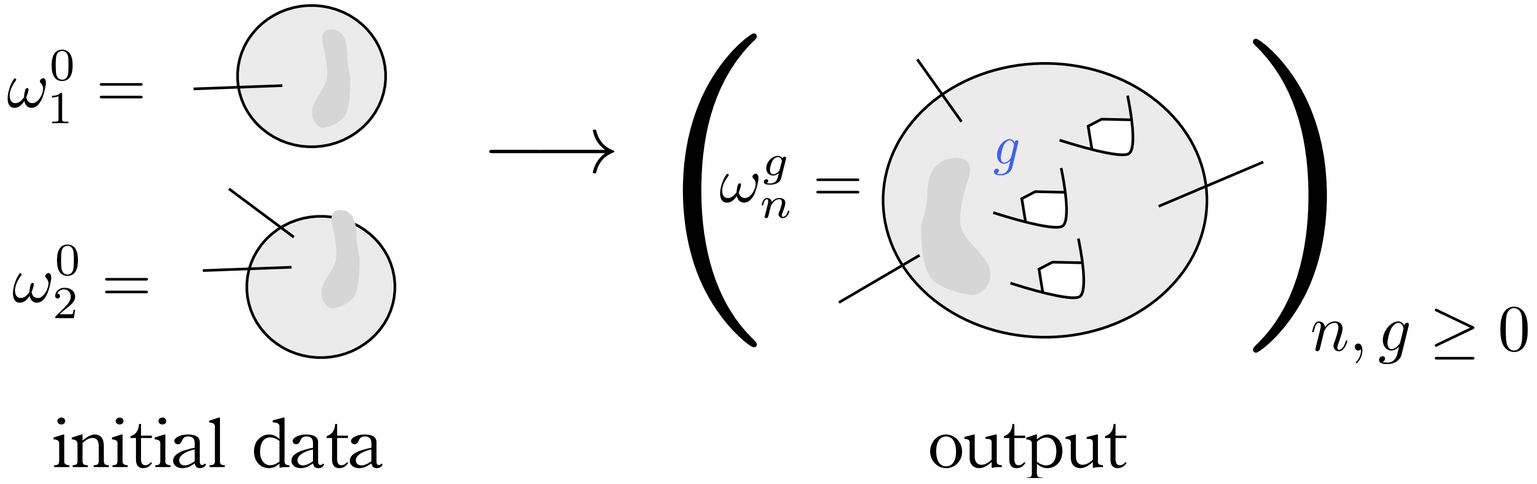
with $\dfrac{1}{2\pi}\mathrm{Re}[uS_{-1}(u)] = \mathrm{Volume}_u(\mathbb{S}_3 \setminus K)$

There are several methods to compute $S_k(u)$

I will present a conjectural one involving topological recursion

# Graph with nodes

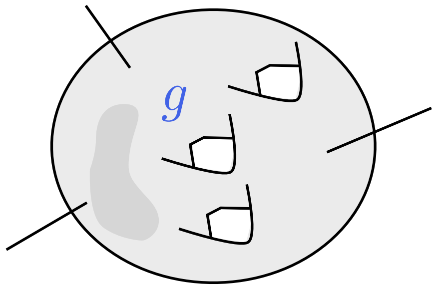
Topological recursion :



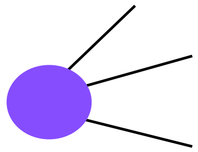
## Graphs with nodes

**Definition**

A graph with nodes is a abstract graph with external legs



- vertices of type 1 are n-valent $(n \geq 1)$
  carry an integer label $g$
  with $\chi = 2g - 2 + n > 0$

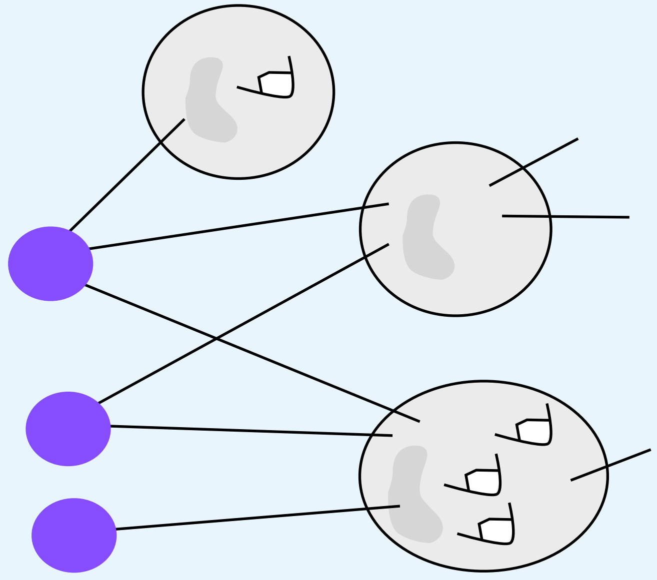


vertices of type 2 (nodes)

- edges : type 1 —— type 2

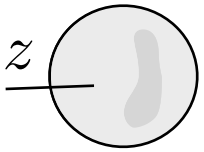
external legs : type 1 ——

Example

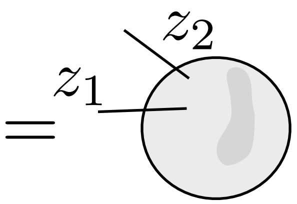


# Graphs with nodes

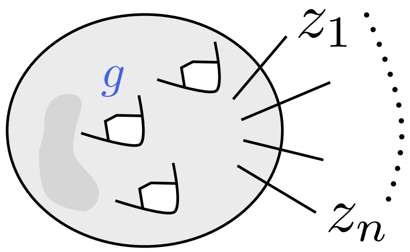
Choose a spectral curve and an initial data $\omega_1^0(z) =$



$\omega_2^0(z_1, z_2) =$



$\xrightarrow{\text{topological recursion}}$ $\omega_n^g(z_1, \ldots, z_n) =$

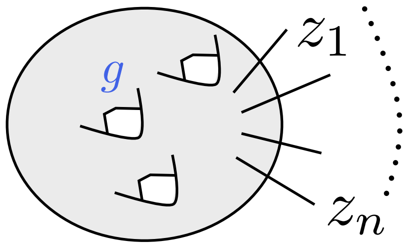


Choose a path $\Gamma$
Choose a cycle $\mathcal{B}$
Choose numbers $(\rho_n)_n$

on the spectral curve

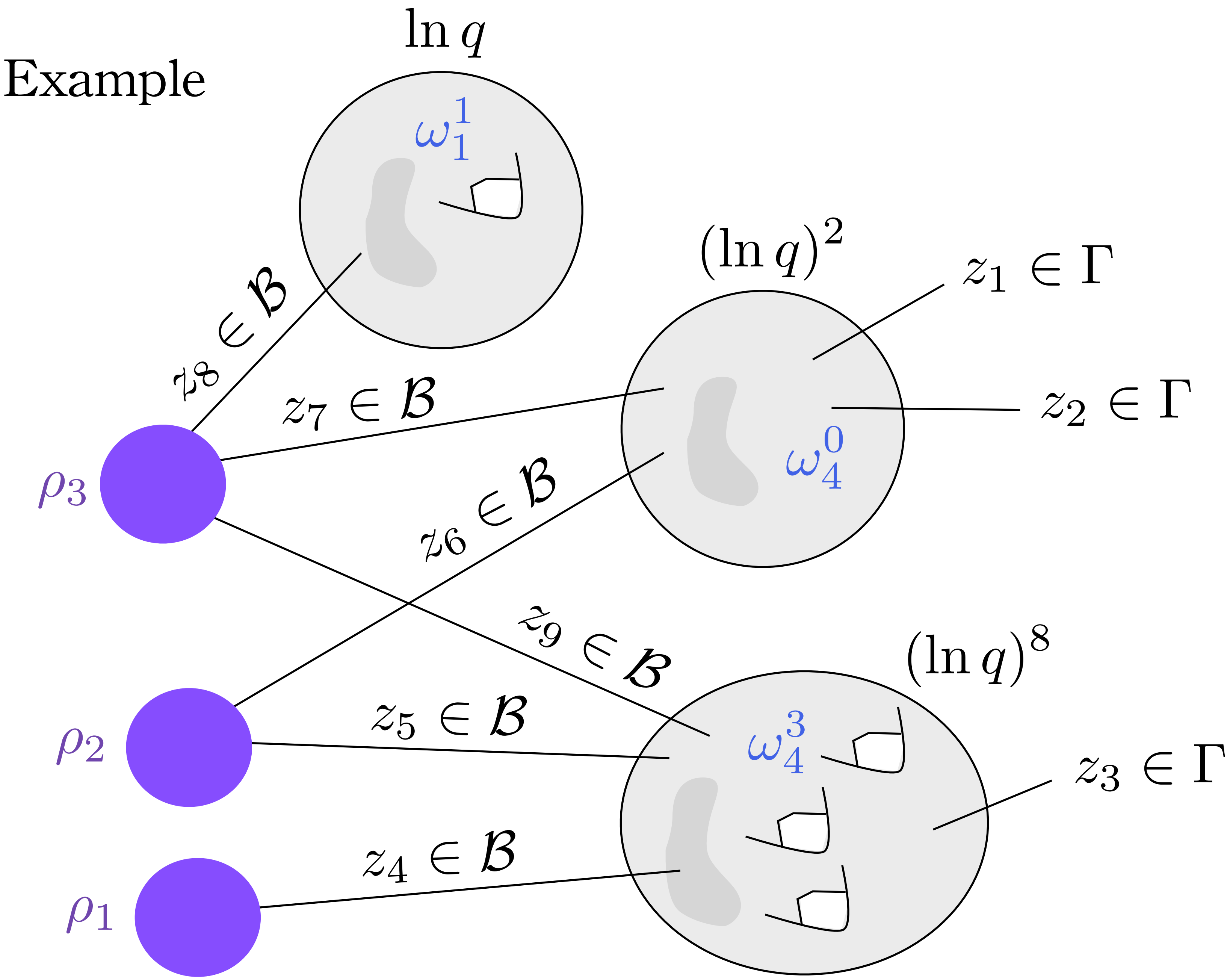
We assign the following weight to a graph with nodes

- assing a variable $z$ to each edge
- local weight for a n-valent vertex of type 1 $(\ln q)^{2g-2+n}$



- local weight for a n-valent vertex of type 2 $\rho_n$
- for external legs, integrate $z \in \Gamma$
- for edges, integrate $z \in \mathcal{B}$
- include the symmetry factor

Example



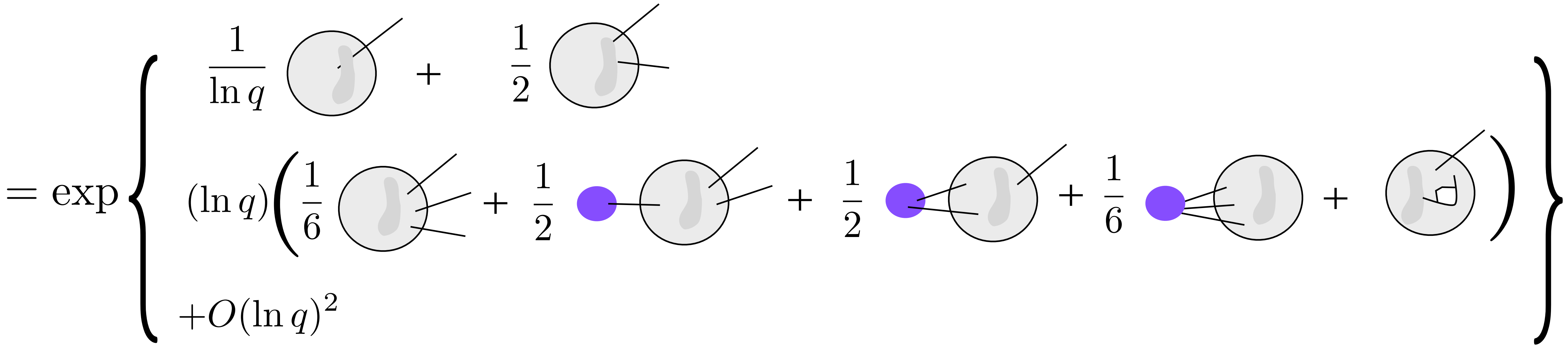
$$\text{weight} = (\ln q)^{11} \rho_1 \rho_2 \rho_3 \Big( \oint_{\mathcal{B}} \omega_1^1 \Big) \Big( \frac{1}{4} \oint_{\mathcal{B}} \oint_{\mathcal{B}} \int_{\Gamma} \int_{\Gamma} \omega_4^0 \Big) \Big( \frac{1}{6} \oint_{\mathcal{B}} \oint_{\mathcal{B}} \oint_{\mathcal{B}} \int_{\Gamma} \omega_4^3 \Big)$$

## Graphs with nodes

We define the wave function as a generating series in $\ln q$

$$\psi(\ln q\,;\,\omega_1^0,\omega_2^0,\Gamma,\mathcal{B},(\rho_n)_n)$$

$$= \exp\left\{\frac{1}{\ln q}\int_\Gamma \omega_1^0 + \frac{1}{2}\int_\Gamma\int_\Gamma \omega_2^0 + \sum \text{weight}\left(\begin{array}{c}\text{connected graphs}\\ \text{with nodes}\end{array}\right)\right\}$$

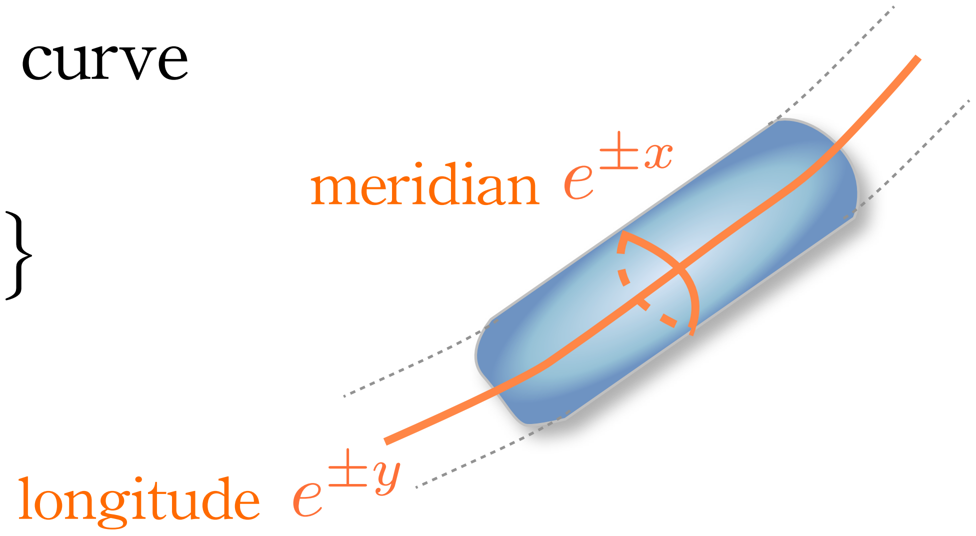


Topological recursion with nodes

# 5. Hyperbolic knots and W(G,ℝ)

## A-polynomial curve

To any knot K , one can associate a spectral curve

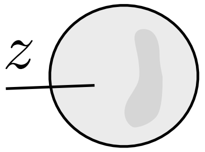


$$\mathcal{C}_K = \{\mathrm{SL}_2(\mathbb{C}) \text{ representations of } \pi_1(\mathbb{S}_3\backslash K)\}$$
$$\simeq \{(x,y)\in\mathbb{C}^2, \qquad A_K(e^x,e^y)=0\}$$

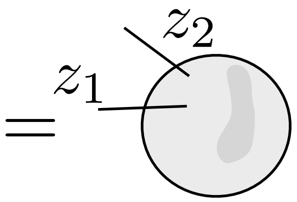
$A_K \in \mathbb{Z}[X,Y]$ is the A-polynomial of K

Cooper, Culler, Gillet, Long, Shalen (94)

We choose the initial data

$\omega_1^0(z) =$


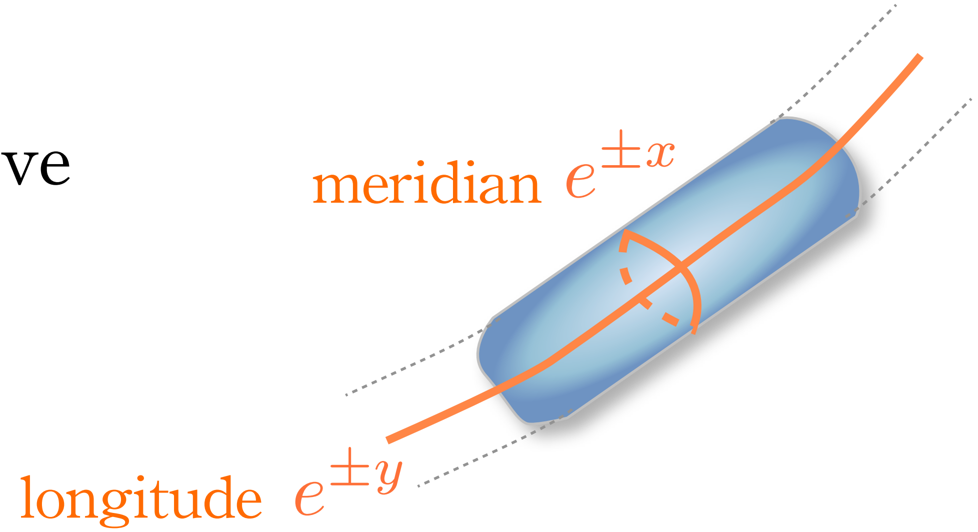
$= y(z)\mathrm{d}x(z)$

$\omega_2^0(z_1,z_2) =$


$= \mathrm{d}_{z_1}\mathrm{d}_{z_2}(\text{Green function}(z_1,z_2) \text{ on } \mathcal{C})$

# 5. Hyperbolic knots and W(G,R) Asymptotics of colored Jones

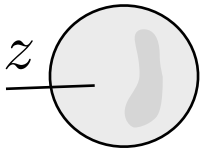
To any knot K , one can associate a spectral curve



$$\mathcal{C}_K = \{(x,y)\in\mathbb{C}^2, \qquad A_K(e^x,e^y)=0\}$$

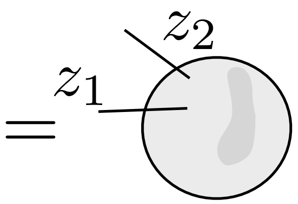
$A_K \in \mathbb{Z}[X,Y]$ is the A-polynomial of K

$\omega_1^0(z) =$



$= y(z)\mathrm{d}x(z)$

$\omega_2^0(z_1,z_2) =$



$= \mathrm{d}_{z_1}\mathrm{d}_{z_2}(\text{Green function}(z_1,z_2) \text{ on } \mathcal{C}_K)$

We choose a path so that $\int_{\Gamma(u)} = \int^{x=u} + \int^{x=-u}$

**Conjecture** Dijkgraaf, Fuji, Manabe (09), corrected by GB, Eynard (12)

For suitable $\mathcal{B}$ and node weights $(\rho_n)_n$

$m \to \infty \quad q \to 1$

$$(J_{m,K}(q))^2 \sim \psi(\ln q\,;\,\omega_1^0,\omega_2^0,\Gamma(u),\mathcal{B},(\rho_n)_n)$$

$u = m\ln(q)$ fixed

# 5. Hyperbolic knots and W(G,R)

Remarks

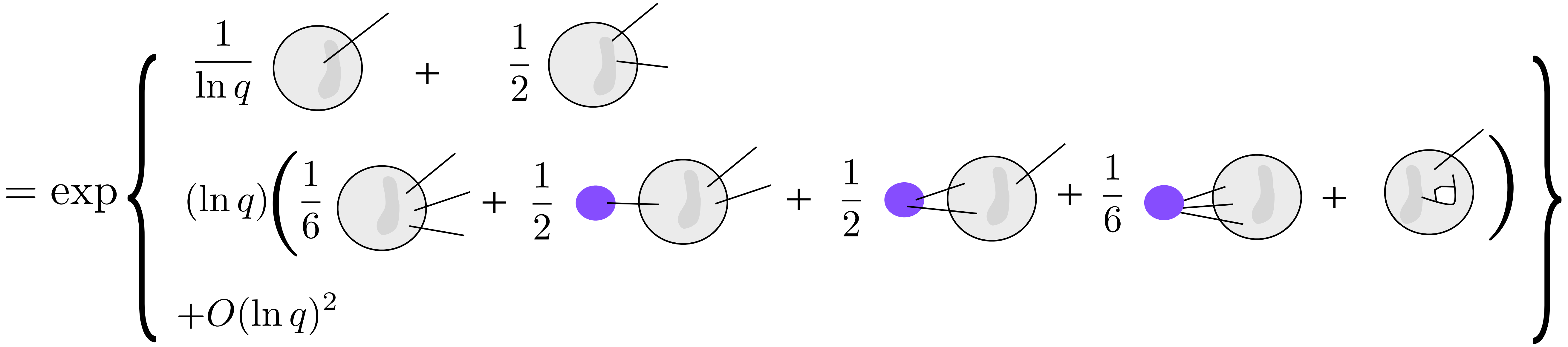
**Conjecture** Dijkgraaf, Fuji, Manabe (09), corrected by B., Eynard (12)

For suitable $\mathcal{B}$ and node weights $(\rho_n)_n$

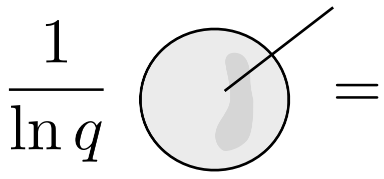
$m \to \infty \quad q \to 1$

$$\left(J_{m,K}(q)\right)^2 \sim \psi(\ln q\,;\omega_1^0,\omega_2^0,\Gamma(u),\mathcal{B},(\rho_n)_n)$$

$u = m\ln(q)$ fixed



- In agreement with the volume conjecture since it is known that



$$= \frac{1}{\ln q}\left(\int^{x=u} + \int^{x=-u} y\,\mathrm{d}x\right) = \frac{m}{2\pi}\mathrm{Volume}_u(\mathbb{S}_3\setminus K)$$

Neumann, Zagier (85)
Yoshida (85)

- genus$(\mathcal{C}) > 0$ for hyperbolic knots $\longrightarrow$ nodes are necessary

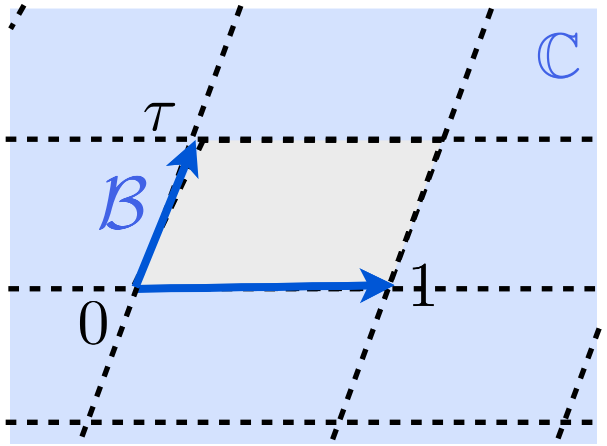
# 5. Hyperbolic knots and W(G,R)

## Example : 8-knot

$$A\,(X,Y) = Y^2X^4 + Y(-X^8+X^6+2X^4+X^2-1)Y + X^4$$

$$X = e^x,\ Y = e^y$$

The curve $A(e^x,e^y)=0$
is $\simeq$ to an elliptic curve $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$



Here is the recipe for the node weights ...

Consider the 3 Jacobi theta series in

$$\begin{cases} \vartheta_2(Q) & = & \sum_{k\in\mathbb{Z}}(-1)^k Q^{k^2/2} \\ \vartheta_3(Q) & = & \sum_{k\in\mathbb{Z}} Q^{k^2/2} \\ \vartheta_4(Q) & = & \sum_{k\in\mathbb{Z}} Q^{(k+1/2)^2/2} \end{cases}$$

It is known that $\frac{(-8\pi^2 Q\partial_Q)^\ell \vartheta_\bullet}{\vartheta_\bullet} = P_\ell(\vartheta_2^4(Q),\vartheta_3^4(Q),E_2(Q))$ where $P_\ell$ is a polynomial

Let us compute $P_\ell(\vartheta_2^4(Q),\vartheta_3^4(Q),0)$ for $Q = e^{2i\pi\tau}$

(= algebraic numbers because A has $\mathbb{Z}$-coefficients)

# 5. Hyperbolic knots and $W(G,\mathbb{R})$

Example : 8-knot

$$A\,(X,Y) = Y^2X^4 + Y(-X^8+X^6+2X^4+X^2-1)Y + X^4$$

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The curve $A(e^x,e^y)=0$
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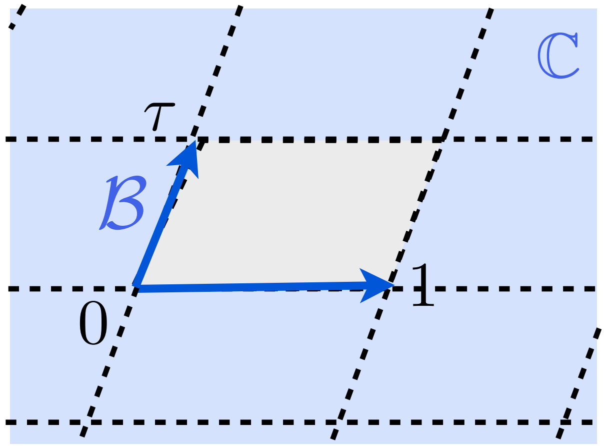


Here is the recipe for the node weights ...

$$\widetilde{\rho}_n = \sum_{(J_i)_i \text{ partition of } n} \prod_i \rho_{|J_i|} \quad \text{and} \quad \widetilde{\rho}_{2n} = \frac{(-8\pi^2 Q\partial_Q)^n\vartheta_\bullet}{\vartheta_\bullet}(Q=e^{2i\pi\tau})\Big|_{E_2\equiv 0}$$

$$\widetilde{\rho}_{2n+1} = 0$$

$D = -8\pi^2\,Q\partial_Q$

$j=2,3,4$ up to permutation

| | | | |
|---|---|---|---|
| $D\vartheta_j/\vartheta_j\big\rvert_{E_2\equiv 0}$ | $\frac{-7+3i\sqrt{15}}{24}$ | $\frac{-7-3i\sqrt{15}}{24}$ | $\frac{7}{12}$ |
| $D^2\vartheta_j/\vartheta_j\big\rvert_{E_2\equiv 0}$ | $\frac{47+21i\sqrt{15}}{96}$ | $\frac{47-21i\sqrt{15}}{96}$ | $-\frac{47}{48}$ |
| $D^3\vartheta_j/\vartheta_j\big\rvert_{E_2\equiv 0}$ | $\frac{-665+9i\sqrt{15}}{1152}$ | $\frac{-665-9i\sqrt{15}}{1152}$ | $-\frac{301}{576}$ |

$\widetilde{\rho}_{2n}$

$$\rho_{2n+1} = 0$$
$$\rho_2 = \frac{7}{12}$$
$$\rho_4 = -2$$
$$\rho_6 = -\frac{511}{576}$$
$$\dots$$

# 5. Hyperbolic knots and $W(G,\mathbb{R})$

## Example : 8-knot

Asymptotics of the colored Jones polynomial

$$J_{K,m}(q) = (\ln q)^{\Delta/2} \exp\left( \sum_{k \geq -1} (\ln q)^k \, S_k(u) + o(\ln q)^{\infty} \right)$$

For the 8-knot, we predict from topological recursion with nodes

$$S_1(u) = -\frac{1}{24\, \sigma^{3/2}(e^u)} (e^{12u} - e^{10u} - 2e^{8u} + 15e^{6u} - 2e^{4u} - e^{2u} + 1)$$

$$S_2(u) = \frac{1}{\sigma^3(e^u)} (e^{12u} - e^{10u} - 2e^{8u} + 5e^{6u} - 2e^{4u} - e^{2u} + 1)$$

$$S_3(u) = \frac{e^{4u}}{180\, \sigma^{9/2}(e^u)} \begin{pmatrix} e^{32u} - 4e^{30u} - 128e^{28u} + 36e^{26u} + 1074e^{24u} - 5630e^{22u} \\ +5782e^{20u} + 7484e^{18u} - 18311e^{16u} + 7484e^{14u} + 5782e^{12u} \\ +1074e^{8u} + 36e^{6u} - 128e^{4u} - 4e^{2u} + 1 \end{pmatrix}$$

where $\sigma(X) = X^8 - 2X^6 - X^4 - 2X^2 + 1$

in agreement with earlier predictions of Dimofte, Gukov, Lenells, Zagier (09)

# Conclusion

The same topological recursion allows to compute

generating series of maps with tubes of any topology ← Tutte eqns.

asymptotic expansion in matrix models ← Schwinger-Dyson eqns.

asymptotic expansion of knot invariants ← ???

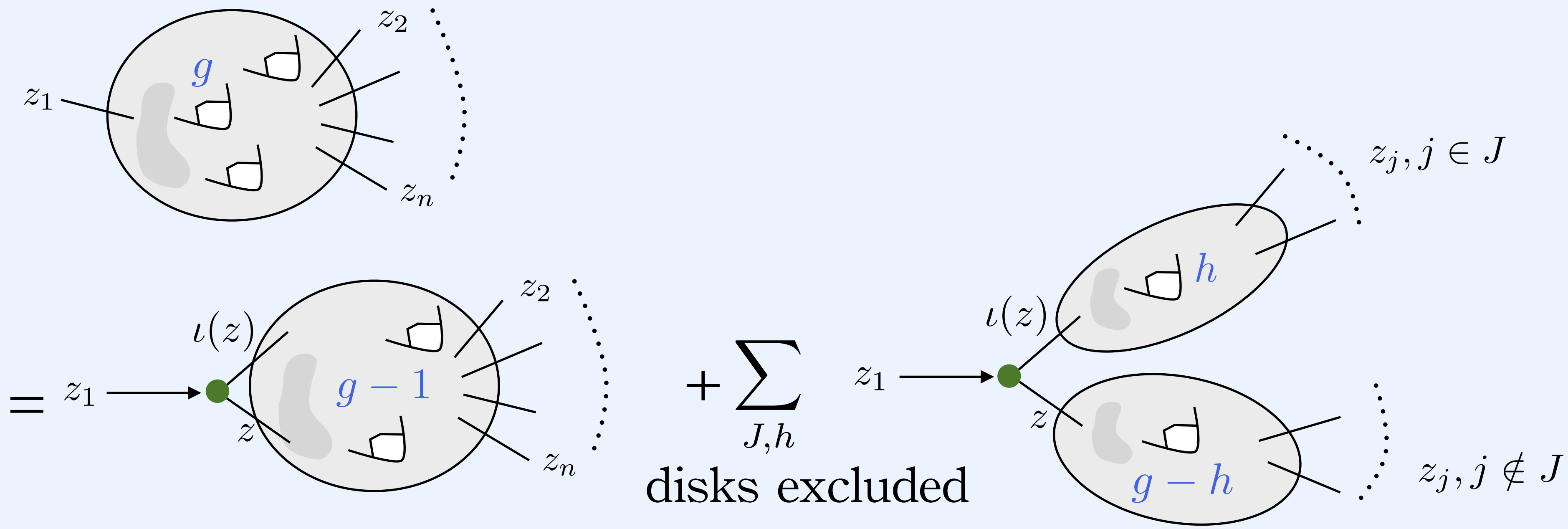
- $G = SU(N)$ $\quad N \to \infty \quad q \to 1$ torus knots
  $R =$ fixed Young tableau $\quad t = N \ln(q)$ fixed

- $G = SU(2)$ $\quad m \to \infty \quad q \to 1$ hyperbolic knots
  $R = \underbrace{\square\square \quad \square}_{(m\ -\ 1)\text{ boxes}}$ $\quad u = m \ln(q)$ fixed

There should be a unifying picture ...

# 2 questions for combinatorists

- Bijection between maps behind the topological recursion ?



- For maps, what would a topological recursion with nodes count ?