Topological recursion: computing asymptotics of knot invariants like counting maps?
Topological recursion: computing asymptotics of knot invariants like counting maps?

1. Maps with(out) tubes
2. ... and formal matrix models
3. Introduction to knot invariants $W(G, R)$
4. Torus knots and $W(G, R)$
5. Hyperbolic knots and $W(G, R)$
1 Maps with(out) tubes ...
A map is a discrete surface obtained by gluing along edges faces with topology of a disk

weight

$t_k$

$(k \geq 1)$

weight per vertex $t$

weight $= t^{11}t_3^3t_4^2t_5t_6^2$
A map with tubes is a discrete surface obtained by gluing along edges faces with topology of a disk

$$\text{weight } t_k$$

$$(k \geq 1)$$

or with topology of a cylinder

$$\text{weight } \gamma t_{k_1,k_2}$$

$$(k_1 + k_2 \geq 1)$$

- weight $t$ per vertex

weight = $t^{22} \gamma^3 t_1 t_2^2 t_4 t_5 t_6 t_{4,8} t_{4,5} t_{6,1}$
Planar maps are those which can be embedded in a sphere.

\[ T_\ell[(t_k)_k; (t_{k_1,k_2})_{k_1,k_2}] = \text{generating series of maps with 1 marked, rooted face of perimeter } \ell \]
1. Maps with(out) tubes

Planar maps with tubes have a nested structure

map with large faces (gasket)

with substitution of tubes (rooted inside)

stuffed with rooted maps with tubes
Planar maps with tubes have a nested structure

$$T_\ell[(t_k)_k; (t_{k_1,k_2})_{k_1,k_2}] = T_\ell[(\tilde{t}_k)_k; 0]$$

with substitution of

$$\tilde{t}_k = t_k + \gamma \sum_m m t_{k,m} T_m[(t_n)_n; (t_{n_1,n_2})_{n_1,n_2}]$$

$\rightarrow$ Counting planar maps with tubes is reduced to counting planar maps
1. Maps with(out) tubes

Definitions

\( t, (t_k)_k \) non-negative is admissible when \( \forall l \geq 1, \ t \partial_t T_{t\ell}[(t_k)_k; 0] < +\infty \)  
(generating series of pointed rooted planar maps)

\( t, (t_k)_k \) real-valued is admissible when \( |t|, (|t_k|)_k \) is admissible

\( t, (t_k)_k, \gamma, t_{k_1, k_2} \) is admissible when \( t, (\tilde{t}_k)_k \) is admissible
1. Maps with(out) tubes

**Definitions**

\[ t,(t_k)_k \text{ non-negative is admissible when } \forall l \geq 1, \quad t\partial_t T_\ell[(t_k)_k; 0] < +\infty \]
(generating series of pointed rooted planar maps)

\[ t,(t_k)_k \text{ real-valued is admissible when } |t|,(|t_k|)_k \text{ is admissible} \]

\[ t,(t_k)_k, \gamma, t_{k_1,k_2} \text{ is admissible when } t,(\tilde{t}_k)_k \text{ is admissible} \]

**Planar 1-cut lemma**  
Bousquet-Mélou (02), GB, Bouttier, Guitter (12)

If \( t,(t_k)_k, \gamma, t_{k_1,k_2} \) is admissible, there exists \( a, b \in \mathbb{R} \)

so that \( W_1^0(x) = \left( \frac{t}{x} + \sum_{\ell \geq 1} \frac{T_\ell[(t_k)_k;(t_{k_1,k_2})_{k_1,k_2}]}{x^{\ell+1}} \right) dx \)

\( \blacklozenge \) is holomorphic in \( \mathbb{C} \setminus [a,b] \)

\( \blacklozenge \) has a discontinuity on \( ]a,b[ \)

\( \blacklozenge \) remains bounded
1. Maps with(out) tubes

\[ x \in \mathbb{C} \setminus [a, b] \]

\[ z \in \mathbb{C} \setminus \overline{D} \]

\[ z \in U \]

conformal mapping

analytic continuation

**Fact** \( \omega^0_1(z) = W^0_1(x(z)) \) can be continued in a neighborhood \( U \) of \( C \)
1. Maps with(out) tubes

Analytic continuation

$x \in \mathbb{C} \setminus [a, b]$  

$z \in \mathbb{C} \setminus \overline{D}$

$z \in U$

conformal mapping  

analytic continuation

**Fact**  

$\omega_1^0(z) = W_1^0(x(z))$ can be continued in a neighborhood $U$ of $C$

We have an involution $z \mapsto \iota(z)$ so that $x(z) = x(\iota(z))$

exchanging interior/exterior of $C$

$W_1^0(x)$ determines a Riemann surface, called spectral curve

on which it can be seen as an analytic 1-form
A map with(out) tubes has genus $g$ if it is connected and can be embedded in

We define the generating series

$$W_n^g(x_1, \ldots, x_n) = \sum_{\ell_1, \ldots, \ell_n \geq 1} \left[ \prod_{i=1}^{n} \frac{dx_i}{x_i^{\ell_i+1}} \right] \times \left\{ \text{generating series of genus } g \text{ maps with rooted marked faces of perimeters } \ell_1, \ldots, \ell_n \right\}$$

and we would like to compute them ...

We cannot speak of nesting for $g > 0$ ! Rather use Tutte bijective decomposition of maps to establish functional relations + a good deal of complex analysis ...
1. Maps with(out) tubes

\[ W_n^g(x_1, \ldots, x_n) = \sum_{\ell_1, \ldots, \ell_n \geq 1} \left[ \prod_{i=1}^{n} \frac{dx_i}{\ell_i + 1} \right] \times \left\{ \text{generating series of genus } g \text{ maps} \right. \\
\left. \text{with rooted marked faces} \right. \\
\left. \text{of perimeters } \ell_1, \ldots, \ell_n \right\} \]

1-cut lemma

If \( t, (t_k)_k, \gamma, t_{k_1, k_2} \) is admissible, then \( W_n^g(x_1, \ldots, x_n) \)

\* is holomorphic in \( \mathbb{C} \setminus [a, b] \)
\* has a discontinuity when \( x_i \in [a, b] \)
\* can be analytically continued to \( \omega_n^g(z_1, \ldots, z_n) \)

on the same Riemann surface

It can be computed by a recursion on \( 2g - 2 + n \)
which takes a universal form

for maps \( \text{Eynard (06)} \)

for maps with tubes \( \text{GB, Eynard, Orantin (13)} \)
It can be computed by a recursion of topological nature.

\[
\omega_n^g(z_1, \ldots, z_n) = \int_{U}^{z_1} \cdots \int_{z_n} \frac{-1}{2} \int_{\iota(z)}^z \omega_2^0(\cdot, z_2) \omega_1^0(z) \omega_1^0(\iota(z)) \cdot \omega_0^0(z_2) \cdot \omega_0^0(\iota(z_2)) \cdots = z_1 \rightarrow \alpha, \beta \rightarrow \omega_0^0(z_2) \cdot \omega_0^0(\iota(z_2)) \cdots
\]

\[
\operatorname{Res}_{z \rightarrow \alpha, \beta} \frac{-1}{2} \int_{\iota(z)}^z \omega_2^0(\cdot, z_2) \omega_1^0(z) \omega_1^0(\iota(z)) \cdots = z_1 \rightarrow \alpha, \beta \rightarrow \omega_0^0(z_2) \cdot \omega_0^0(\iota(z_2)) \cdots
\]

\[
= z_1 \rightarrow \alpha, \beta \rightarrow \omega_0^0(z_2) \cdot \omega_0^0(\iota(z_2)) \cdots + \sum_{J, h} z_1 \rightarrow \alpha, \beta \rightarrow \omega_0^0(z_2) \cdot \omega_0^0(\iota(z_2)) \cdots \text{ disks excluded} \]

\[
= z_1 \rightarrow \alpha, \beta \rightarrow \omega_0^0(z_2) \cdot \omega_0^0(\iota(z_2)) \cdots + \sum_{J, h} z_1 \rightarrow \alpha, \beta \rightarrow \omega_0^0(z_2) \cdot \omega_0^0(\iota(z_2)) \cdots \text{ disks excluded} \]

\[
= z_1 \rightarrow \alpha, \beta \rightarrow \omega_0^0(z_2) \cdot \omega_0^0(\iota(z_2)) \cdots + \sum_{J, h} z_1 \rightarrow \alpha, \beta \rightarrow \omega_0^0(z_2) \cdot \omega_0^0(\iota(z_2)) \cdots \text{ disks excluded} \]
\( \omega_{n}^{g} \) can be computed by a universal recursion of topological nature if one knows \( \omega_{1}^{0} = \) and \( \omega_{2}^{0} = \)

but the combinatorial interpretation of the recursion is not known ...

\( \omega_{n}^{g}(\nu(z), z_{2}, \ldots, z_{n}) \)

**Definition** The topological recursion is the algorithm \( n, g \geq 0 \)

\[ \omega_{1}^{0} = \]
\[ \omega_{2}^{0} = \]

initial data

output
... and formal matrix models

Relation to maps
Multidimensional integrals ...
... and their asymptotics
2. Formal matrix models

Generating series of maps can be represented by formal matrix integrals

Brézin, Itzykson, Parisi, Zuber (78)

if we introduce

the generating series of faces

$$D(x) = \frac{N}{t} \sum_k \frac{t_k}{k} x^k$$

the gaussian measure on

$$N \times N$$ hermitian matrices

$$d\mu(M) = dM \ e^{-N \text{Tr}M^2/2t}$$

then, the formal series in $$t, t_k$$ has a well-defined decomposition

$$\mu \left[ e^{\text{Tr} D(M)} \prod_{i=1}^n \text{Tr} \frac{dx_i}{x_i - M} \right]_c = \sum_{g \geq 0} N^{2-2g-n} \ W^g_n (x_1, \ldots, x_n)$$

generating series of maps

of genus $$g$$ with $$n$$ rooted marked faces
2. Formal matrix models

Generating series of maps with tubes have a similar representation reformulation of Gaudin, Mehta, Kostov (90s)

if we introduce the generating series of faces

with topology of a disk \( D(x) = \frac{N}{t} \sum_k \frac{t_k}{k} x^k \)

with topology of a cylinder \( C(x, y) = \frac{\gamma}{t} \sum_{k_1, k_2} \frac{t_{k_1, k_2}}{2k_1 k_2} x^{k_1} y^{k_2} \)

then, the formal series in \( t, t_k, \gamma, t_{k_1, k_2} \) has a well-defined decomposition

\[
\mu \left[ e^{\text{Tr} D(M) + \text{Tr} C(M \otimes 1_N, 1_N \otimes M)} \prod_{i=1}^n \text{Tr} \frac{dx_i}{x_i - M} \right] c = \sum_{g \geq 0} N^{2-2g-n} W_n^g (x_1, \ldots, x_n)
\]

generating series of maps with tubes of genus \( g \) with \( n \) rooted marked faces... and maps with tubes
2. Formal matrix models

U(N) invariant measures:

the integrals over M reduces to an integral over its eigenvalues

e.g. the partition function

\[
\mu \left[ e^{\text{Tr} \, D(M) + \text{Tr} \, C(M \otimes 1_N, 1_N \otimes M)} \right] = \frac{\text{Vol}(U(N))}{N!(2\pi)^N} \int_{\mathbb{R}^N} \prod_{i=1}^{N} d\lambda_i e^{-\frac{N\lambda_i^2}{2t} + D(\lambda_i) + C(\lambda_i, \lambda_i)} \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)^2 e^{2C(\lambda_i, \lambda_j)}
\]

arbitrary two-point interaction
vanishing like a square
at short distance

as presented for formal integrals
A summary on the large $N$ expansion of

$$\int \prod_{i=1}^{N} d\lambda_i e^{-NV(\lambda_i)} \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)^2 K(\lambda_i, \lambda_j)$$

K non-vanishing

formal integral (\rightarrow combinatorics, 1/N expansion, 1-cut lemma)

$K = 1$  Ambjørn, Makeenko, Chekhov, Kristjansen (90s), Eynard (04)

$K \neq 1$  GB, Eynard, Orantin (13)

cv integral + 1-cut + assumptions \rightarrow 1/N expansion

$K = 1$  Albeverio, Pastur, Shcherbina (00), Ercolani, McLaughlin (04)

GB, Guionnet (11)

$K \neq 1$  GB, Guionnet, Kozlowski (in progress)

cv integral + several cuts + assumptions \rightarrow 1/N expansion

$K = 1$  heuristics : Bonnet, David, Eynard (00), Eynard (07)

proof : GB, Guionnet (13)

$K \neq 1$  GB, Guionnet, Kozlowski (in progress)

Topological recursion

Topological recursion with nodes
(see later ...)
3 Introduction to knots

Definition, classification
Knot invariants
Asymptotics and why should we care?
A **knot** is an isotopy class of proper embedding of a circle in $S^3$.
Knots are complicated (as much as arithmetic in $\mathbb{Z}$ is ...)

There exist infinitely many prime knots, which fall in 3 families

- $(P,Q)$ torus knots

- hyperbolic knots $\mathbb{S}_3 \setminus K$ admits a complete hyperbolic metric

- satellite knots (uncharted territory)
Knots are complicated (as much as arithmetic in $\mathbb{Z}$ is ...)

Hard algorithmic problems:

- unknotting number
- distinguishing two knots

$\longrightarrow$ construction of (computable) knot invariants to give partial answers

For any compact Lie group $G$, irreducible representation $\mathcal{R}$, one can construct an invariant using representations of quantum groups

$$K \longmapsto W_{k}(G, \mathcal{R}; q) \in \mathbb{Z}[q, q^{-1}]$$

knot colored HOMFLY polynomial

behaving nicely under geometric operations (gluings, cabling, ...)
3. Introduction to knots

Knot invariants

For any compact Lie group $G$, irreducible representation $\mathcal{R}$, one can construct an invariant using representations of quantum groups

$$K \mapsto W_k(G, \mathcal{R}; q) \in \mathbb{Z}[q, q^{-1}]$$

colored HOMFLY polynomial

behaving nicely under geometric operations (gluings, cabling, ...)

$G = SU(2)$

$m$-th colored Jones polynomial

$$\mathcal{R} = \begin{array}{c} \square \\ \square \ldots \square \end{array}$$

$(m - 1)$ boxes

$$J^\text{unknot}_m(q) = \frac{q^m - q^{-m}}{q - q^{-1}}$$

$$J^{8-knot}_m(q) = \frac{q^m - q^{-m}}{q - q^{-1}} \left( \sum_{k=0}^{m-1} (q^2)_k (1/q^2)_k \right)$$

but in general no closed formulas ...
3. Introduction to knots

Asymptotics of knot invariants

Enumeration of knots by matrix model techniques
Zinn-Justin, Zuber

has nothing to do with the topic of this talk

For a given knot $\mathcal{K}$
we would like to compute the asymptotic expansion of $W_k(G,R;q)$ when

$\bullet \ G = SU(N)$
$R = \text{fixed Young tableau}$

$N \to \infty \quad q \to 1$
$t = N \ln(q) \text{ fixed}$

(Theory of LMO invariants)

$\bullet \ G = SU(2)$
$R = \begin{array}{c}
\Box \\
\Box \\
\Box \\
\end{array} \cdots \Box$

$m \to \infty \quad q \to 1$
$u = m \ln(q) \text{ fixed}$

$(m - 1) \text{ boxes}$
3. Introduction to knots

For a given knot $K$
we would like to compute the asymptotic expansion of $W_k(G,R;q)$ when

1. If $G = SU(N)$
   $R = \text{fixed Young tableau}$
   $t = N \ln(q) \text{ fixed}$

   (theory of LMO invariants)

2. Theorem: for torus knots, by the topological recursion

   - $G = SU(2)$
     $R = \begin{array}{cccc}
     \cdot & \cdot & \cdot & \cdot
     \end{array}$
     ($m - 1$) boxes

   $m \to \infty \quad q \to 1$
   $u = m \ln(q) \text{ fixed}$

3. Conjecture: for hyperbolic knots, by the topological recursion
   with nodes
The interest about asymptotics of knot invariants started from

\* \( G = \text{SU}(2) \)

\( R = \square \cdots \square \)  \((m - 1) \) boxes

\( m \to \infty \quad q \to 1 \)

\( u = m \ln(q) \) fixed

**Volume conjecture**  
Kashaev (98), H. Murakami (00)

If \( K \) is a hyperbolic knot,  

\[
\lim_{m \to \infty} \frac{2\pi}{m} \ln \left| J_{K,m}(q = e^{2i\pi/m}) \right| = \text{Volume}(S^3 \setminus K)
\]

Algebraic construction  \( \longrightarrow \)  Geometric information
3. Introduction to knots

Why should we care?

The interest about asymptotics of knot invariants started from

\[ \mathbf{G} = \text{SU}(2) \]

\[ \mathbf{R} = \begin{array}{|c|c|} \hline & \\
\hline & \\
\hline \end{array} \ldots \begin{array}{|c|} \rule{0pt}{2pt} \\
\hline \end{array} \]

(m - 1) boxes

\[ m \to \infty \quad q \to 1 \]

\[ u = m \ln(q) \quad \text{fixed} \]

Volume conjecture

Kashaev (98), H. Murakami (00)

If \( K \) is a hyperbolic knot, \( \lim_{m \to \infty} \frac{2\pi}{m} \ln |J_{K,m}(q = e^{2i\pi/m})| = \text{Volume}(\mathbb{S}_3 \setminus K) \)

Other conjectures

other values of \( u \) and asymptotic expansion

Gukov (04)

arithmeticity, modularity, ...

Zagier et al. (09)

Computation and unified understanding of those properties?

Relation to other fields (counting surfaces ...)?
4 Torus knots and $w(G, r)$

Definition, classification

Knot invariants

Asymptotics and why should we care?

(There is a pun hidden in this slide)
4. Torus knots and $W(G,R)$

$W_k(G,R;q)$

can be computed as an observable in Chern-Simons theory in $S_3$  
Witten (89)

CS theory = quantum field theory with measure

$$d\mu_{CS}[A] = DA \exp \left\{ - \frac{1}{\ln q} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \right\}$$

$A$ = section of a $G$ principal bundle over $S_3$

Fact  
If $C$ is a loop in $S_3$

$$\mu \left[ \text{Tr}_R(\text{holonomy of } A \text{ along } C) \right]$$

is a topological invariant

and coincides with $W_k(G,R;q)$
For torus knots, the path integral reduces to a finite dimensional integral!
(exact saddle point)  Rozansky (98), Mariño (02), Beasley, Witten (07), Källen (09)

\[ G = SU(N) \]
\[ \ell = N \ln(q) \]
\[ K = \]

\[ d\mu^P_Q = \prod_{i=1}^{N} d\lambda_i e^{-N\lambda_i^2/2PQt} \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)^2 K(\lambda_i, \lambda_j) \]

with  \[ K(x, y) = \frac{\sinh(x-y/2P)}{(x-y/2P)} \frac{\sinh(x-y/2Q)}{(x-y/2Q)} \]

and the knot invariants are  \[ W(SU(N), R; q) = \frac{\mu^P_Q \left[ \text{Schur}_R(e^{\lambda_1}, \ldots, e^{\lambda_N}) \right]}{\mu^P_Q[1]} \]
4. Torus knots and $W(G,R)$

The case of torus knots

\[ d\mu_N^{P,Q} = \prod_{i=1}^N d\lambda_i \, e^{-N\lambda_i^2/2PQ} \prod_{1\leq i<j\leq N} (\lambda_i - \lambda_j)^2 K(\lambda_i, \lambda_j) \]

with \( K(x, y) = \frac{\sinh(\frac{x-y}{2P})}{(\frac{x-y}{2P})} \frac{\sinh(\frac{x-y}{2Q})}{(\frac{x-y}{2Q})} \)

and the knot invariants are \( W(SU(N), R; q) = \frac{\mu_N^{P,Q}[\text{Schur}_R(e^{\lambda_1}, \ldots, e^{\lambda_N})]}{\mu_N^{P,Q}[1]} \)

It is convenient to use another basis of symmetric functions

\( \text{Schur}_R \longleftrightarrow \text{power sums} \)

and form a generating series of expectation values of power sums

\[ W_n(x_1, \ldots, x_n) = \frac{\mu_N^{P,Q}\left[ \prod_{i=1}^n \text{Tr} \frac{dx_i}{x_i-e^{-M/PQ}} \right]_c}{\mu_N^{P,Q}[1]} \quad M = \text{diag}(e^{\lambda_1}, \ldots, e^{\lambda_N}) \]
4. Torus knots and $W(G, r)$

$\text{SU}(N)$ at large $N$

$$d\mu_{N}^{P, Q} = \prod_{i=1}^{N} d\lambda_i e^{-N\lambda_i^2/2PQt} \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)^2 K(\lambda_i, \lambda_j)$$

with $K(x, y) = \frac{\sinh(x-y)}{(2P)} \frac{\sinh(x-y)}{(2Q)}$

We would like to compute $W_n(x_1, \ldots, x_n) = \frac{\mu_N^{P, Q} \left[ \prod_{i=1}^{n} \text{Tr} \frac{dx_i}{x_i - e^{M/PQ}} \right]_c}{\mu_N^{P, Q}[1]}$

**Theorem**

- prediction, check: Brini, Eynard, Mariño (11)
- proof: GB, Eynard, Orantin (13)

Assume $t = N \ln q > 0$ fixed

- There is a large $N$ expansion $W_n = \sum_{g \geq 0} N^{2-2g-n} W_n^g$
- There is an explicit formula for $W_1^0$ and $W_2^0$
- $W_n^g$ have $1$-cut $[a, b]$, and are computed by the topological recursion
Reminder: topological recursion

It can be computed by a recursion of topological nature.

\[
\omega_n^g(z_1, \ldots, z_n) = \left. -\frac{1}{2} \int \omega_0^0(\cdot, z_2) \right|_{z \rightarrow \alpha, \beta} \omega_2^0(z) - \omega_1^0(\iota(z))
\]

\[
\operatorname{Res} \left. \frac{\Delta}{\iota(z)} \omega_1^0(z) - \omega_1^0(\iota(z)) \right|_{z \rightarrow \alpha, \beta}
\]

\[
\sum_{J, h} (g - h) \left. \omega_1^0(z) \right|_{z \rightarrow \alpha, \beta}
\]

\[
\omega_2^0(z) \left. \omega_2^0(\cdot, z_2) \right|_{z \rightarrow \alpha, \beta}
\]

the involution \( \iota \)

\( z \in U \)

\( z_j, j \in J \)

\( z_j, j \notin J \)

\( g - 1 \)

\( g - h \)

\( \iota(z) \)

\( z \)

\( z_1 \)

\( z_2 \)

\( z_n \)

\( \iota(z) \)

\( z_1 \)

\( z_2 \)

\( z_n \)

\( \iota(z) \)

\( z_1 \)

\( z_2 \)

\( z_n \)
4. Torus knots and \( W(G, r) \)

Torus knots: conclusion

\[
d\mu_N^{P,Q} = \prod_{i=1}^N d\lambda_i \, e^{-N\lambda_i^2/2PQ}\, t \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)^2 K(\lambda_i, \lambda_j)
\]

with \( K(x, y) = \frac{\sinh\left(\frac{x-y}{2P}\right)}{\left(\frac{x-y}{2P}\right)} \frac{\sinh\left(\frac{x-y}{2Q}\right)}{\left(\frac{x-y}{2Q}\right)} \)

We would like to compute \( W_n(x_1, \ldots, x_n) = \frac{\mu_N^{P,Q} \left[ \prod_{i=1}^n \text{Tr} \frac{dx_i}{x_i - e^{M/PQ}} \right] c}{\mu_N^{P,Q}[1]} \)

• This model is a special case of
  the matrix model enumerating maps with tubes

any explanation ??

• Cannot be generalized yet to hyperbolic knots ...
  (no finite-dimensional reduction of CS theory)
5 Hyperbolic knots and $W(g,R)$

Definition, classification

Knot invariants

Asymptotics and why should we care?
The interest about asymptotics of knot invariants started from

\[ G = SU(2) \quad \text{and} \quad R = \Box \cdots \Box \quad \text{(m - 1) boxes} \]

\[ m \to \infty \quad q \to 1 \]

\[ u = m \ln(q) \quad \text{fixed} \]

**Volume conjecture**  
Kashaev (98), H. Murakami (00)

If $K$ is a hyperbolic knot,

\[ \lim_{m \to \infty} \frac{2\pi}{m} \ln \left| J_{K,m}(q = e^{2i\pi/m}) \right| = \text{Volume}(S_3 \setminus K) \]
The interest about asymptotics of knot invariants started from

\[ G = SU(2) \]

\[ m \to \infty \quad q \to 1 \]

\[ u = m \ln(q) \quad \text{fixed} \]

---

**Generalized ...**

Gukov (04)

If \( K \) is a hyperbolic knot,

\[
J_{k,m}(q) = (\ln q)^{\Delta/2} \exp \left( \sum_{k \geq -1} (\ln q)^k S_k(u) + o(\ln q)^\infty \right)
\]

with

\[
\frac{1}{2\pi} \text{Re}[uS_{-1}(u)] = \text{Volume}_u(S_3 \setminus K)
\]

There are several methods to compute \( S_k(u) \)

I will present a conjectural one involving topological recursion
5. Hyperbolic knots and $w(g, R)$

Graph with nodes

Topological recursion:

$\omega_1^0 = \begin{array}{c}
\begin{array}{c}
\text{initial data}
\end{array}
\end{array}$

$\omega_2^0 = \begin{array}{c}
\begin{array}{c}
\text{output}
\end{array}
\end{array}$

$\begin{pmatrix}
\omega_n^g = \begin{array}{c}
\begin{array}{c}
\text{output}
\end{array}
\end{array}, n, g \geq 0
\end{pmatrix}$
A graph with nodes is a abstract graph with external legs

- vertices of type 1 are n-valent \((n \geq 1)\)
  - carry an integer label \(g\)
  - with \(\chi = 2g - 2 + n > 0\)

- vertices of type 2 (nodes)

- edges: type 1 —— type 2

- external legs: type 1 ——

Example
Choose a spectral curve and an initial data

\[ \omega_1^0(z) = \frac{z}{z} \quad \omega_2^0(z_1, z_2) = \frac{z_1}{z_2} \]

Topological recursion

\[ \omega_n^g(z_1, \ldots, z_n) = \]

Choose a path \( \Gamma \) on the spectral curve
Choose a cycle \( B \) on the spectral curve
Choose numbers \( (\rho_n)_n \)

We assign the following weight to a graph with nodes

- assigning a variable \( z \) to each edge
- local weight for a n-valent vertex of type 1 \((\ln q)^{2g-2+n}\)
- local weight for a n-valent vertex of type 2 \(\rho_n\)
- for external legs, integrate \( z \in \Gamma \)
- for edges, integrate \( z \in B \)
- include the symmetry factor
Example

\[
\text{weight} = (\ln q)^{11} \rho_1 \rho_2 \rho_3 \left( \int_B \omega_1 \right) \left( \frac{1}{4} \int_B \int_B \int_B \int_B \omega_1 \right) \left( \frac{1}{6} \int_B \int_B \int_B \int_B \omega_3 \right)
\]
We define the wave function as a generating series in $\ln q$

$$\psi(\ln q ; \omega_1^0, \omega_2^0, \Gamma, B, (\rho_n)_n) = \exp \left\{ \frac{1}{\ln q} \int_\Gamma \omega_1^0 + \frac{1}{2} \int_\Gamma \int_\Gamma \omega_2^0 + \sum \text{weight} \left( \begin{array}{c} \text{connected graphs} \\ \text{with nodes} \end{array} \right) \right\}$$

$$= \exp \left\{ \frac{1}{\ln q} \left( \begin{array}{c} \frac{1}{6} \\ + \frac{1}{2} \end{array} \right) + \frac{1}{2} \left( \begin{array}{c} \frac{1}{6} \\ + \frac{1}{2} \end{array} \right) + \frac{1}{2} \left( \begin{array}{c} \frac{1}{6} \\ + \frac{1}{6} \end{array} \right) + \right\} + O(\ln q)^2$$

Topological recursion with nodes
5. Hyperbolic knots and $W(g,R)$

A-polynomial curve

To any knot $\mathcal{K}$, one can associate a spectral curve

$$\mathcal{C}_\mathcal{K} = \{\text{SL}_2(\mathbb{C}) \text{ representations of } \pi_1(S_3 \setminus \mathcal{K})\}$$

$$\simeq \{(x, y) \in \mathbb{C}^2, A_{\mathcal{K}}(e^x, e^y) = 0\}$$

$A_{\mathcal{K}} \in \mathbb{Z}[X,Y]$ is the A-polynomial of $\mathcal{K}$

Cooper, Culler, Gillet, Long, Shalen (94)

We choose the initial data

$$\omega_1^0(z) = z = y(z)\, dx(z)$$

$$\omega_2^0(z_1, z_2) = z_1z_2 = d_{z_1}d_{z_2} (\text{Green function}(z_1, z_2) \text{ on } \mathcal{C})$$
To any knot $\mathcal{K}$, one can associate a spectral curve

$$\mathcal{C}_\mathcal{K} = \{(x, y) \in \mathbb{C}^2, \quad A_\mathcal{K}(e^x, e^y) = 0\}$$

$A_\mathcal{K} \in \mathbb{Z}[X, Y]$ is the A-polynomial of $\mathcal{K}$

$$\omega^0_1(z) = \frac{z}{2} = y(z) \text{d}x(z)$$

$$\omega^0_2(z_1, z_2) = \frac{z_1 z_2}{2} = d_{z_1} d_{z_2} \text{(Green function}(z_1, z_2) \text{ on } \mathcal{C}_\mathcal{K})$$

We choose a path so that $\int_{\Gamma(u)} = \int^{x=u} + \int^{x=-u}$

**Conjecture** Dijkgraaf, Fuji, Manabe (09), corrected by GB, Eynard (12)

For suitable $\mathcal{B}$ and node weights $(\rho_n)_n$

$$(J_{m, \mathcal{K}}(q))^2 \sim \psi(\ln q; \omega^0_1, \omega^0_2, \Gamma(u), \mathcal{B}, (\rho_n)_n)$$

$m \to \infty \quad q \to 1$

$u = m \ln(q)$ fixed
Conjecture
Dijkgraaf, Fuji, Manabe (09), corrected by B., Eynard (12)

For suitable $\mathcal{B}$ and node weights $(\rho_n)_n$

$$(J_{m,\kappa}(q))^2 \sim \psi(\ln q; \omega^0_1, \omega^0_2, \Gamma(u), \mathcal{B}, (\rho_n)_n)$$

$m \to \infty \quad q \to 1$

$u = m \ln(q)$ fixed

$$\exp \left\{ \frac{1}{\ln q} \left( \frac{1}{6} \right) + \frac{1}{2} \left( \right) + \frac{1}{2} \left( \right) + \frac{1}{6} \left( \right) \right\} + O(\ln q)^2$$

• In agreement with the volume conjecture since it is known that

$$\frac{1}{\ln q} = \frac{1}{\ln q} \left( \int_{x=-u}^{x=u} y \, dx \right) = \frac{m}{2\pi} \text{Volume}_u(S_3 \setminus \mathcal{K})$$

Neumann, Zagier (85)
Yoshida (85)

• genus$(\mathcal{C}) > 0$ for hyperbolic knots $\rightarrow$ nodes are necessary
5. Hyperbolic knots and $W(g, R)$

Example: 8-knot

$$A(X, Y) = Y^2X^4 + Y(-X^8 + X^6 + 2X^4 + X^2 - 1)Y + X^4$$

$X = e^x$, $Y = e^y$

The curve $A(e^x, e^y) = 0$

is $\simeq$ to an elliptic curve $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$

Here is the recipe for the node weights ...

Consider the 3 Jacobi theta series in

$$\begin{align*}
\vartheta_2(Q) &= \sum_{k \in \mathbb{Z}} (-1)^k Q^{k^2/2} \\
\vartheta_3(Q) &= \sum_{k \in \mathbb{Z}} Q^{k^2/2} \\
\vartheta_4(Q) &= \sum_{k \in \mathbb{Z}} Q^{(k+1/2)^2/2}
\end{align*}$$

It is known that $\frac{(-8\pi^2 Q \partial_Q)^\ell \vartheta_4}{\vartheta_4} = P_\ell(\vartheta_2^4(Q), \vartheta_3^4(Q), E_2(Q))$ where $P_\ell$ is a polynomial

Let us compute $P_\ell(\vartheta_2^4(Q), \vartheta_3^4(Q), 0)$ for $Q = e^{2i\pi \tau}$

($= \text{algebraic numbers because } A \text{ has } \mathbb{Z}-\text{coefficients}$)
5. Hyperbolic knots and $W(G, R)$

Example: 8-knot

$A(X, Y) = Y^2 X^4 + Y(-X^8 + X^6 + 2X^4 + X^2 - 1)Y + X^4$

$X = e^x$, $Y = e^y$

The curve $A(e^x, e^y) = 0$ is $\simeq$ to an elliptic curve $\mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z})$

Here is the recipe for the node weights ...

\[
\tilde{\rho}_n = \sum_{(J_i)_i \text{ partition of } n} \prod_i \rho_{|J_i|} \quad \text{and} \quad \tilde{\rho}_{2n} = \left. \frac{(-8\pi^2 Q \partial Q)^n \vartheta}{\vartheta} (Q = e^{2i\pi \tau}) \right|_{E_2 \equiv 0}
\]

$\tilde{\rho}_{2n+1} = 0$

$D = -8\pi^2 Q \partial Q$

$j = 2, 3, 4$ up to permutation

\[
\begin{array}{ccc}
D\vartheta_j/\vartheta_j \big|_{E_2 \equiv 0} & -7+3i\sqrt{15} & -7-3i\sqrt{15} \\
& \frac{24}{24} & \frac{24}{24} \\

D^2\vartheta_j/\vartheta_j \big|_{E_2 \equiv 0} & 47+21i\sqrt{15} & 47-21i\sqrt{15} \\
& \frac{96}{96} & \frac{96}{96} \\

D^3\vartheta_j/\vartheta_j \big|_{E_2 \equiv 0} & -665+9i\sqrt{15} & -665-9i\sqrt{15} \\
& \frac{1152}{1152} & \frac{1152}{1152} \\
\end{array}
\]

$\rho_{2n} = \frac{7}{12}$

$\rho_2 = \frac{7}{12}$

$\rho_4 = -2$

$\rho_6 = -\frac{511}{576}$

$\vdots$
Asymptotics of the colored Jones polynomial

\[ J_{k,m}(q) = (\ln q)^{\Delta/2} \exp \left( \sum_{k \geq -1} (\ln q)^k S_k(u) + o(\ln q)^\infty \right) \]

For the 8-knot, we predict from topological recursion with nodes

\[ S_1(u) = -\frac{1}{24 \sigma^{3/2}(e^u)} (e^{12u} - e^{10u} - 2e^{8u} + 15e^{6u} - 2e^{4u} - e^{2u} + 1) \]

\[ S_2(u) = \frac{1}{\sigma^3(e^u)} (e^{12u} - e^{10u} - 2e^{8u} + 5e^{6u} - 2e^{4u} - e^{2u} + 1) \]

\[ S_3(u) = \frac{e^{4u}}{180 \sigma^{9/2}(e^u)} \left( e^{32u} - 4e^{30u} - 128e^{28u} + 36e^{26u} + 1074e^{24u} - 5630e^{22u} + 5782e^{20u} + 7484e^{18u} - 18311e^{16u} + 7484e^{14u} + 5782e^{12u} + 1074e^{8u} + 36e^{6u} - 128e^{4u} - 4e^{2u} + 1 \right) \]

where \( \sigma(X) = X^8 - 2X^6 - X^4 - 2X^2 + 1 \)

in agreement with earlier predictions of Dimofte, Gukov, Lenells, Zagier (09)
Conclusion

The same topological recursion allows to compute

- generating series of maps with tubes of any topology \( t = N \) \( \rightarrow \infty \) \( q \rightarrow 1 \) \( \text{torus knots} \)
- asymptotic expansion in matrix models \( G = SU(N) \) \( N \rightarrow \infty \) \( q \rightarrow 1 \) \( \text{Schwinger-Dyson eqns.} \)
- asymptotic expansion of knot invariants \( G = SU(2) \) \( m \rightarrow \infty \) \( q \rightarrow 1 \) \( \text{hyperbolic knots} \)

\* \( G = SU(N) \)
\* \( G = SU(2) \)

\( R = \) fixed Young tableau
\( R = \begin{array}{ccc} \, & \, & \, \\ \, & \, & \, \end{array} \) \( m \rightarrow \infty \) \( u = m \ln(q) \) fixed

\( (m - 1) \) boxes

There should be a unifying picture ...
2 questions for combinatorists

• Bijection between maps behind the topological recursion?

\[ g - 1 = z_1 + \sum_{J, h} z_1^g - h \text{ disks excluded} \]

\[ z_j, j \in J \]

\[ z_j, j \notin J \]

• For maps, what would a topological recursion with nodes count?

\[ \sum \]