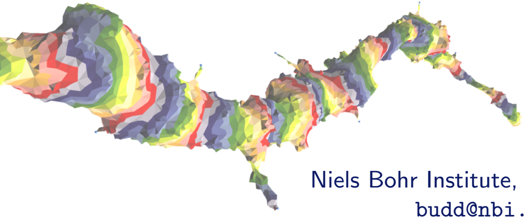
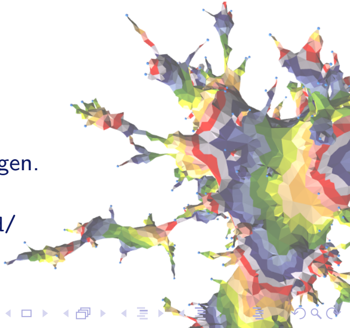
Journées Cartes,
Jun. 20, 2013

From planar maps to spatial topology change in 2d gravity

Timothy Budd

(joint work with J. Ambjørn)





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Outline

- Introduction to the (generalized) CDT model of 2d gravity
- Bijection between labeled quadrangulations and labeled planar maps
- Generalized CDT solved in terms of labeled trees
- Two-point functions
- Bijection between pointed quadrangulations and pointed planar maps
- Loop identity of generalized CDT

Causal Dynamical Triangulations (CDT) [Ambjørn, Loll, '98]

- CDT in 2d is a statistical system with partition function

$$Z_{CDT} = \sum_{\mathcal{T}} \frac{1}{C_{\mathcal{T}}} g^{N(\mathcal{T})}$$

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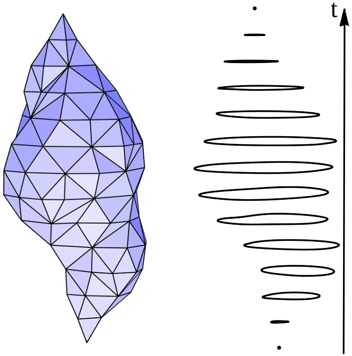

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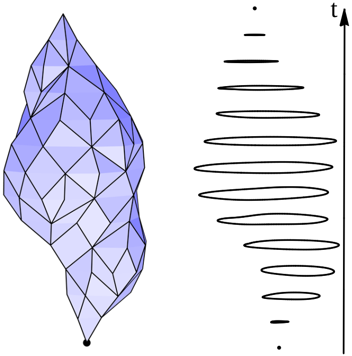

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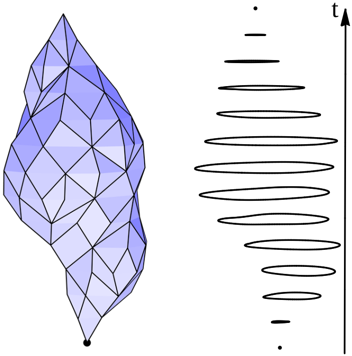
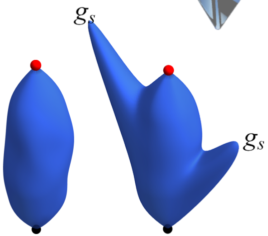
- What if we allow more than one local maximum?



Generalized CDT

- Allow spatial topology to change in time.
  Assign a coupling g_s to each baby universe.



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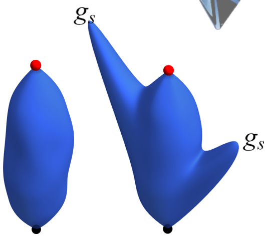

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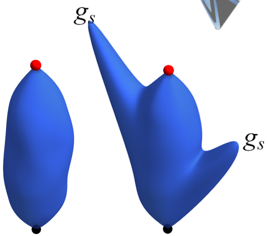

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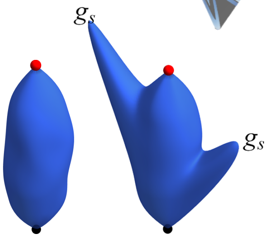

- Generalized CDT partition function

$$Z(g, \mathfrak{g}) = \sum_{\mathcal{Q}} \frac{1}{C_{\mathcal{Q}}} g^{N} \mathfrak{g}^{N_{max}},$$

sum over quadrangulations $\mathcal{Q}$ with N faces, a marked origin, and N_{max} local maxima of the distance to the origin.

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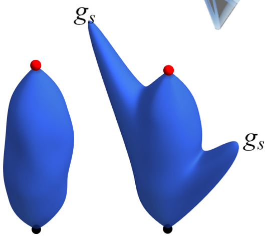

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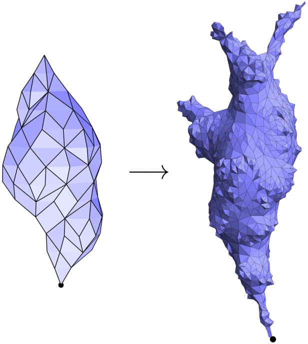
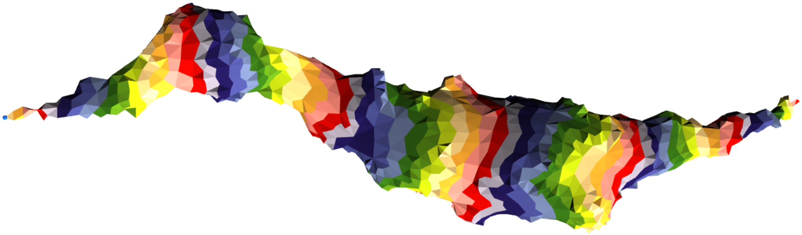
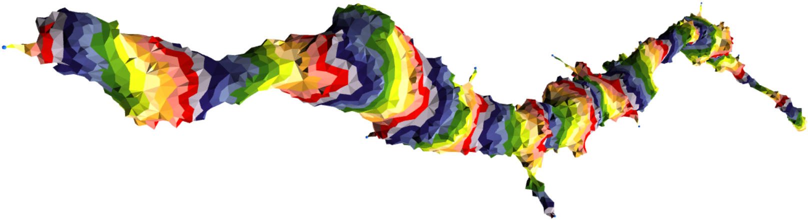


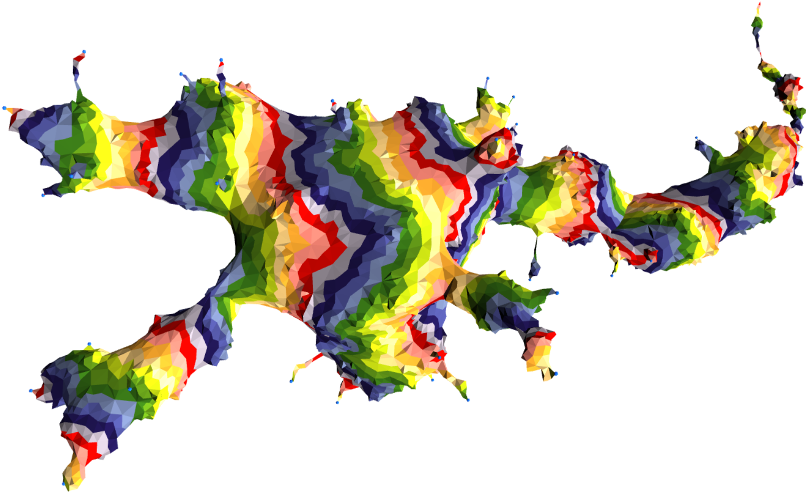
$$N = 2000,\ \mathfrak{g} = 0,\ N_{max} = 1$$



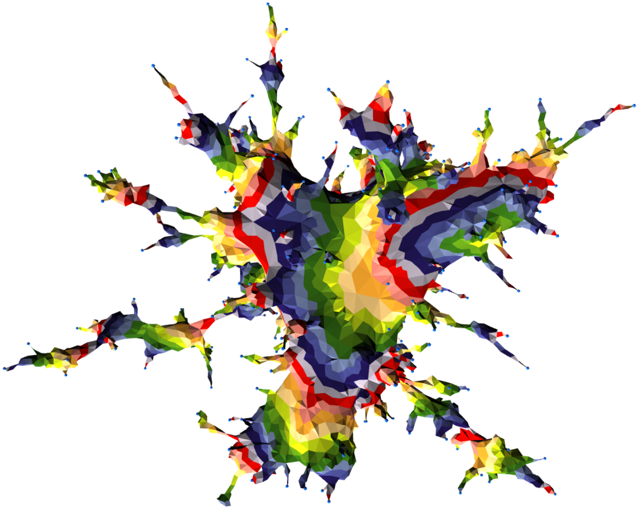
$N = 5000,\ \mathfrak{g} = 0.00007,\ N_{max} = 12$



$N = 7000$, $\mathfrak{g} = 0.0002$, $N_{max} = 38$

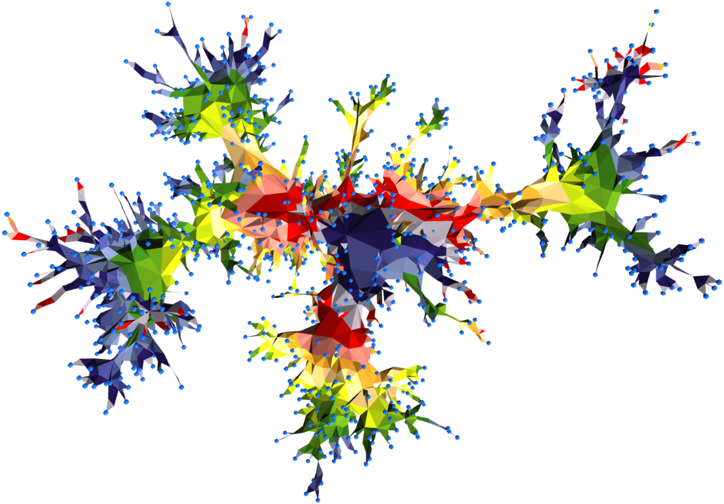


$N = 7000,\ \mathfrak{g} = 0.004,\ N_{max} = 221$

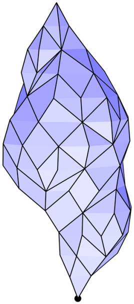


$N = 4000,\ \mathfrak{g} = 0.02,\ N_{max} = 362$

$N = 2500,\ \mathfrak{g} = 1,\ N_{max} = 1216$



Causal triangulations and trees



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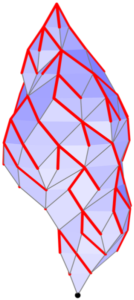

- Union of all left-most geodesics running *away from* the origin.

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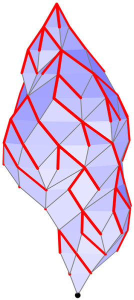

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- Simple enumeration of planar trees:

$\#\Big\{$  $\Big\}_N = C(N), \quad C(N) = \frac{1}{N+1}\binom{2N}{N}$

[Malyshev, Yambartsev, Zamyatin '01]

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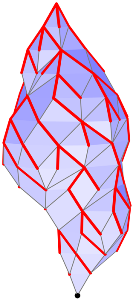

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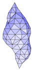
$\#\Big\{$  $\Big\}_N = C(N), \quad C(N) = \frac{1}{N+1}\binom{2N}{N}$

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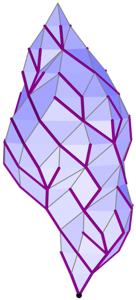

- Union of all left-most geodesics running *towards* the origin.

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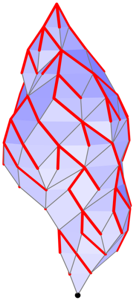

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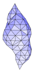
$\#\Big\{$  $\Big\}_N = C(N), \quad C(N) = \frac{1}{N+1}\binom{2N}{N}$

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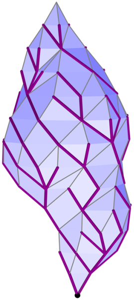
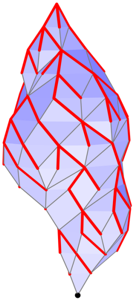
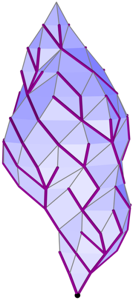


- Union of all left-most geodesics running *towards* the origin.
- Both generalize to generalized CDT leading to different representations.

Causal triangulations and trees





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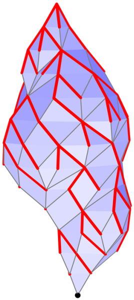
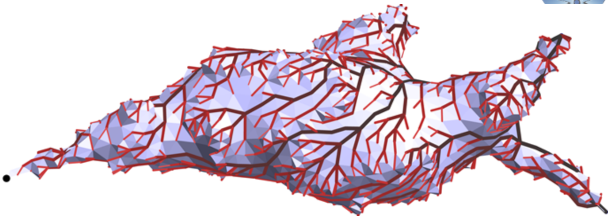
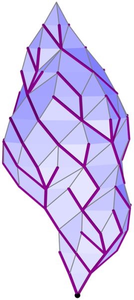




- Labeled planar trees: Schaeffer's bijection.



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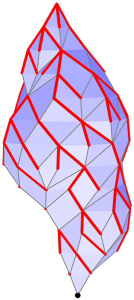

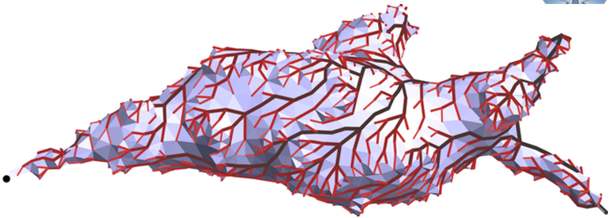

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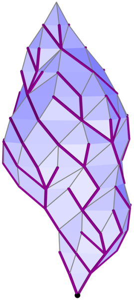

- Unlabeled planar maps (one face per local maximum).

Labeled quadrangulations

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Schaeffer's rules define a map $\Psi : \mathcal{Q}^{(l)} \to \mathcal{M}^{(l)}$:

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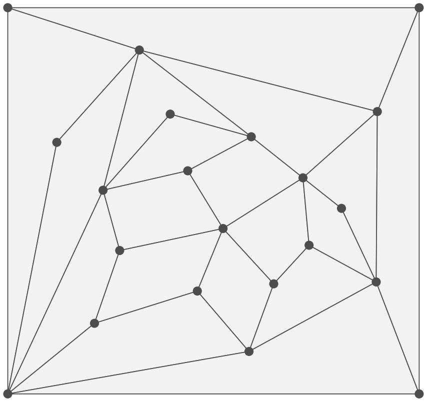

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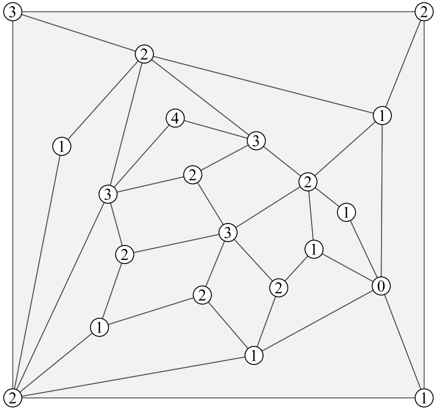

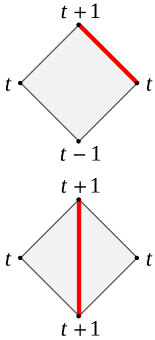

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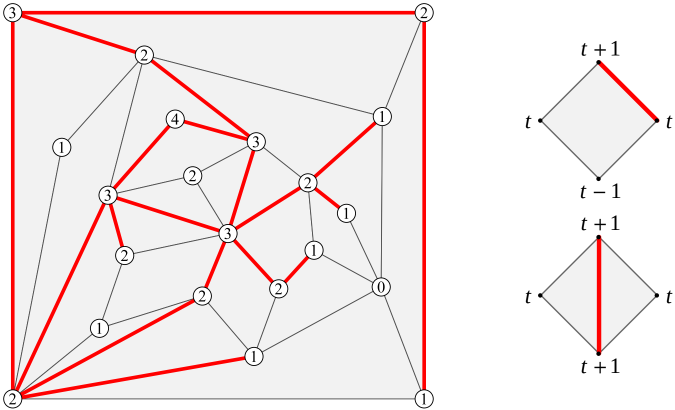

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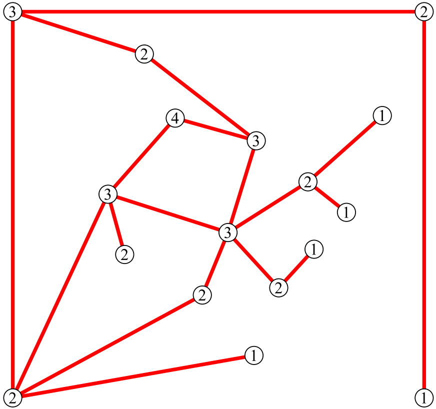

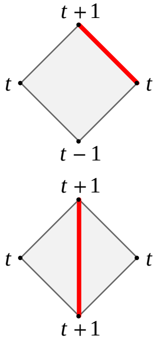

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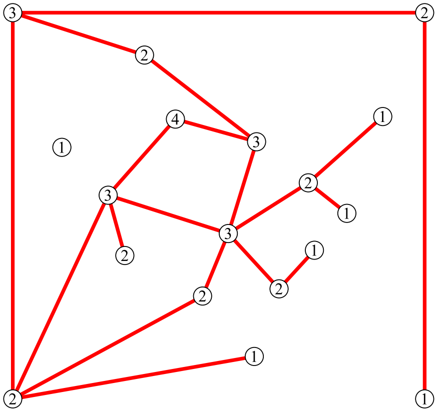

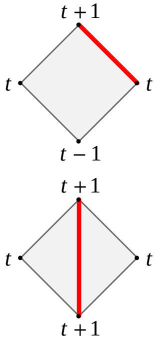

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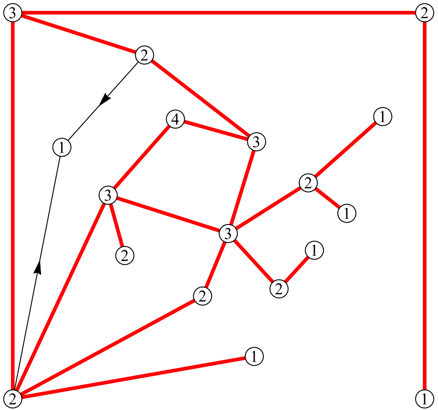

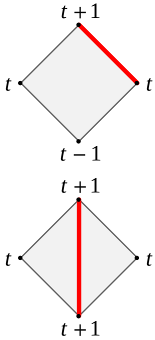

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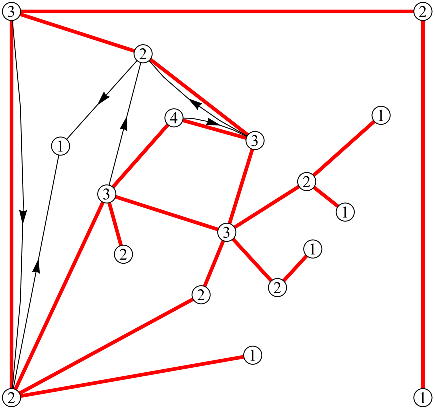

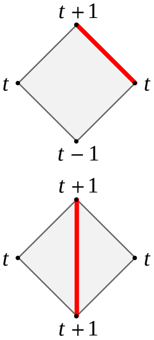

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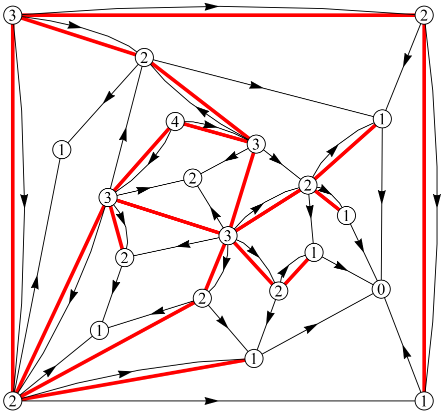

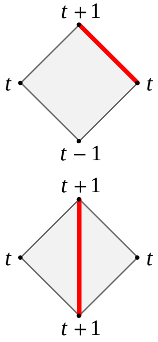

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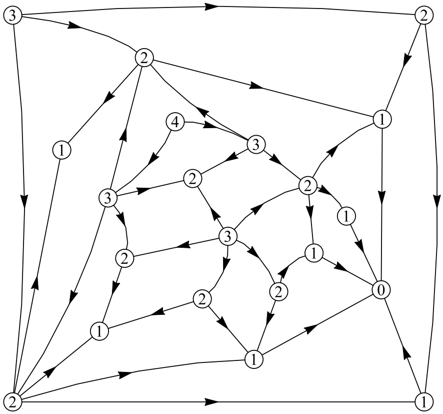

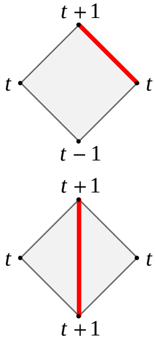

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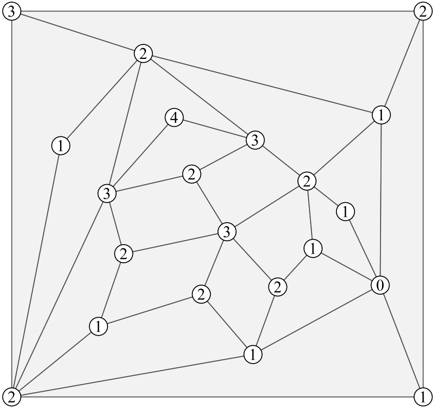

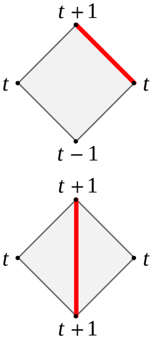

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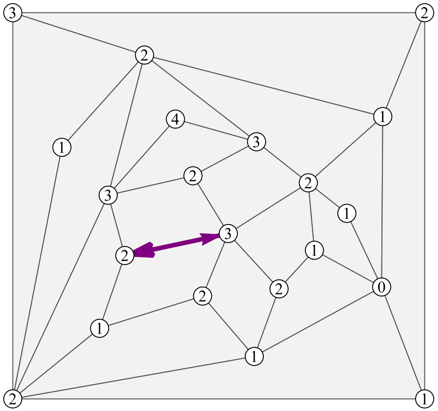

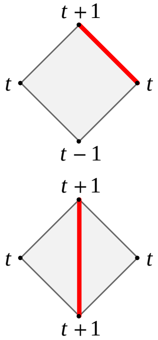

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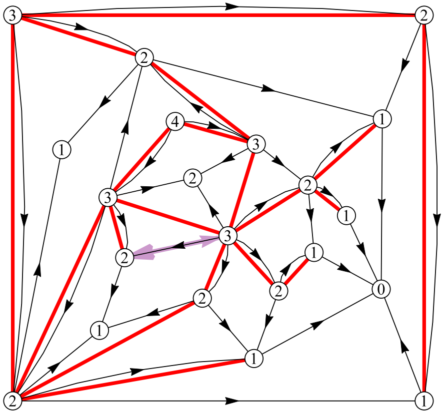

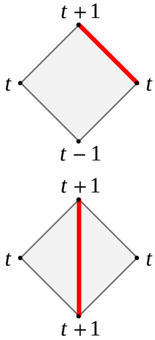

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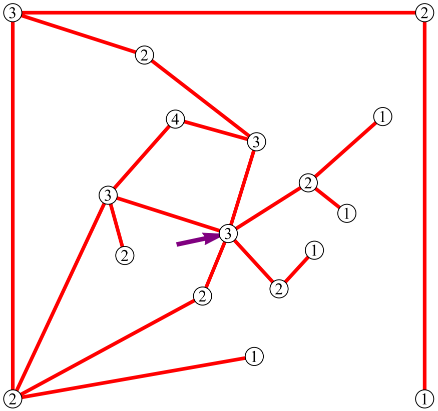

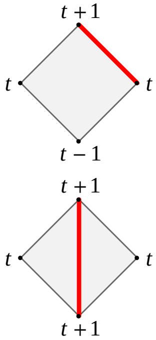

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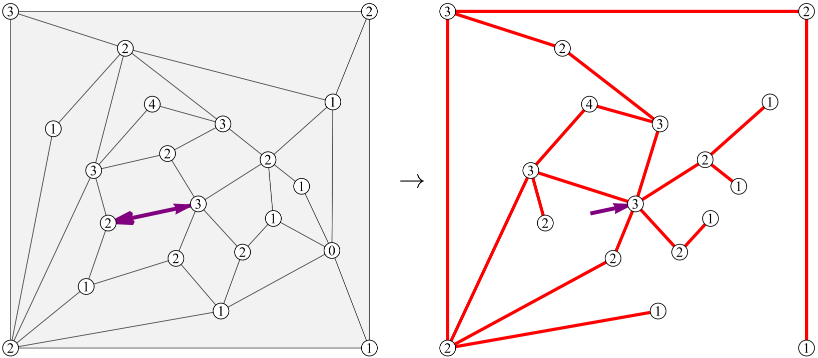

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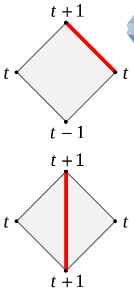

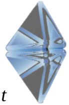

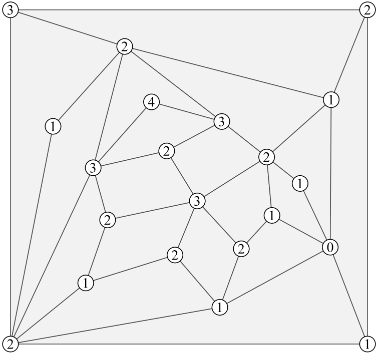

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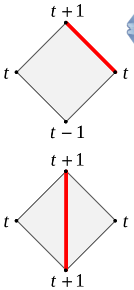

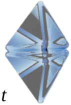

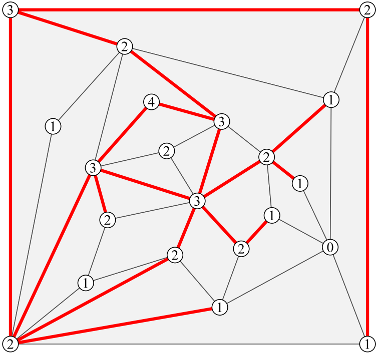

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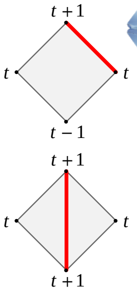

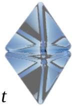

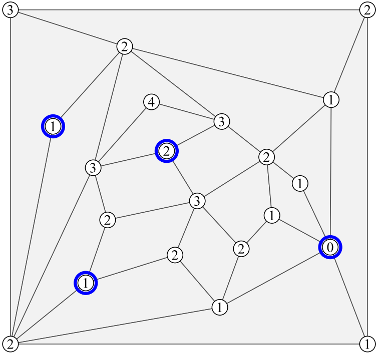

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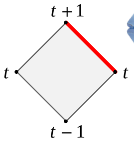

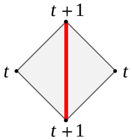

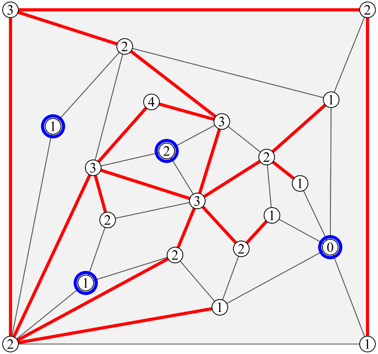

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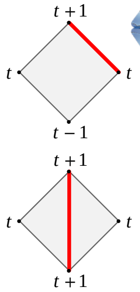

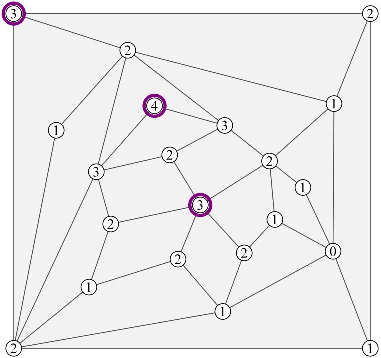

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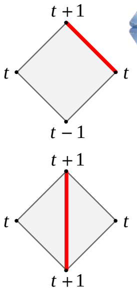

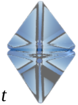

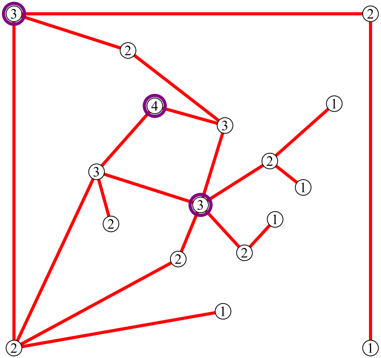

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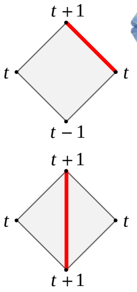

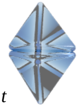

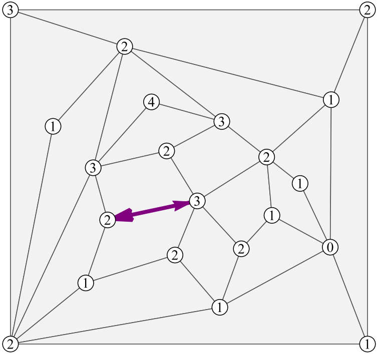

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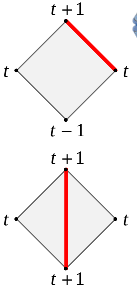

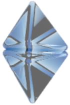

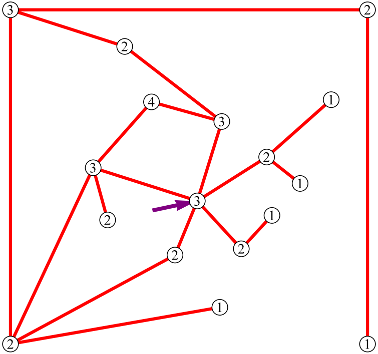

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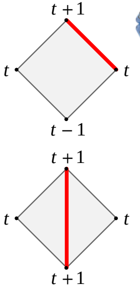

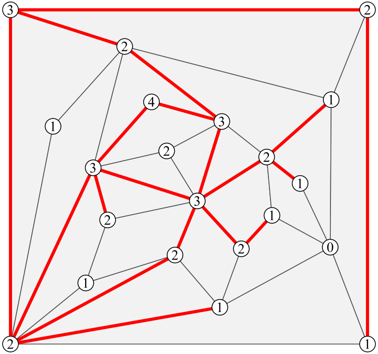

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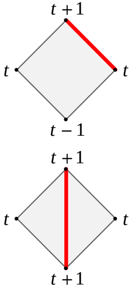

Theorem 1$^-$

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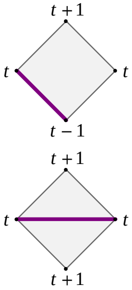

Cori–Vauquelin–Schaeffer bijection [Cori, Vauquelin, ’81][Schaeffer, ’98]

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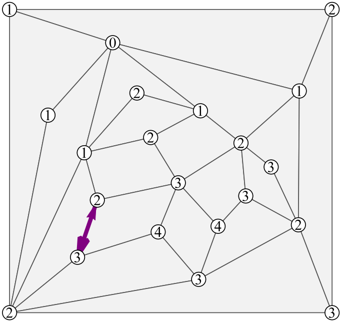

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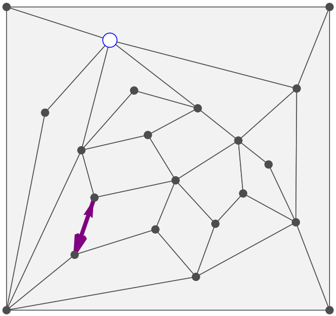

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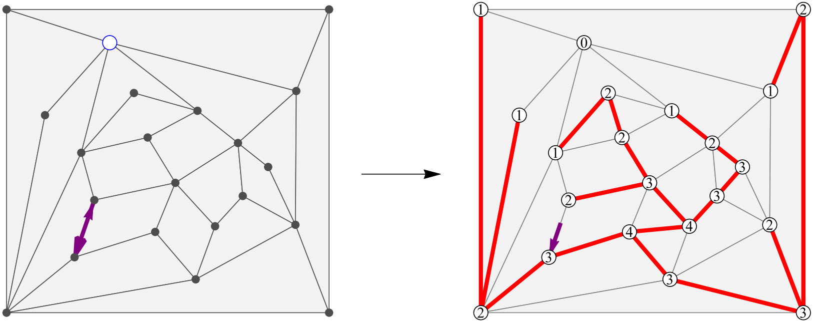

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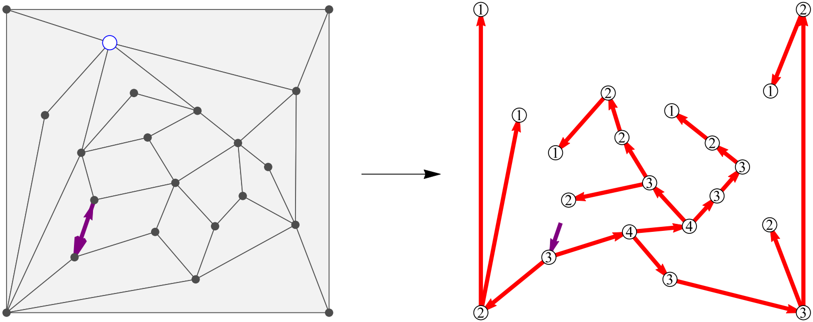

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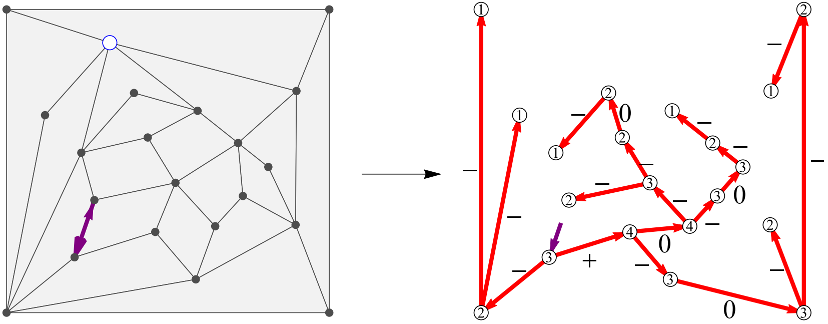

Cori–Vauquelin–Schaeffer bijection [Cori, Vauquelin, '81][Schaeffer, '98]

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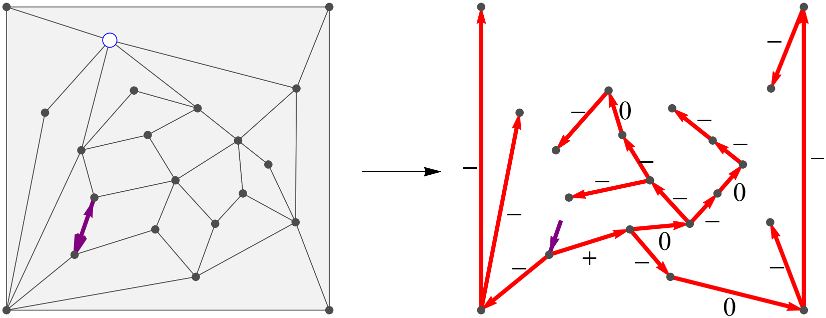

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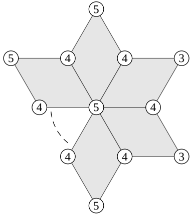

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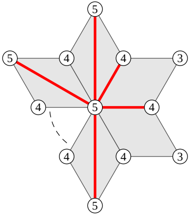

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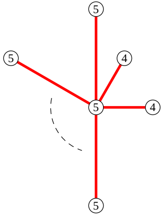

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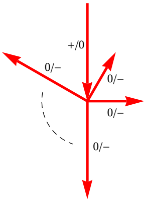

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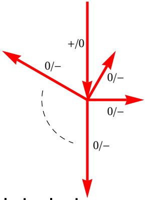

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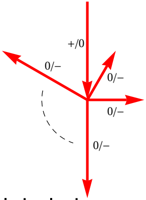

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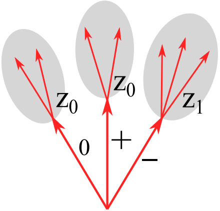
- Satisfy recursion relations:

$$z_1 = \sum_{k=0}^{\infty} (z_1 + z_0 + z_0)^k g^k = (1 - g z_1 - 2 g z_0)^{-1}$$

$$z_0 = \sum_{k=0}^{\infty} (z_1 + z_0 + z_0)^k g^k + (\mathfrak{g} - 1) \sum_{k=0}^{\infty} (z_1 + z_0)^k g^k$$

$$= z_1 + (\mathfrak{g} - 1)(1 - g z_1 - g z_0)^{-1}$$



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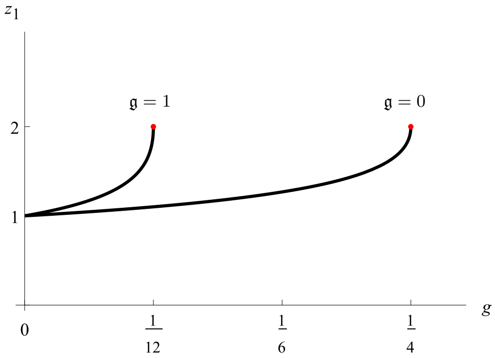

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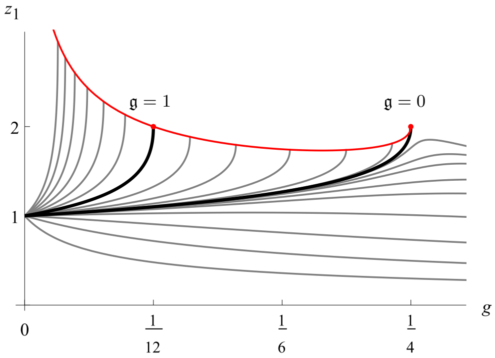

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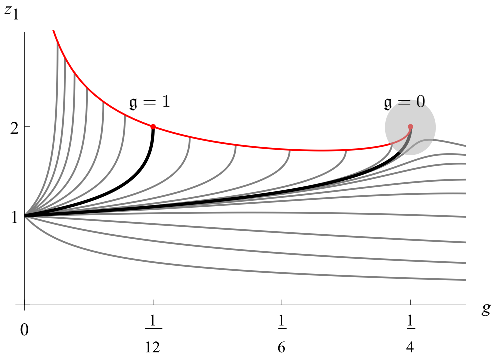

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$$\frac{\langle N_{max}(\mathfrak{g})\rangle_N}{N} = 2\left(\frac{\mathfrak{g}}{2}\right)^{2/3} + \mathcal{O}(\mathfrak{g}), \quad \frac{\langle N_{max}(\mathfrak{g}=1)\rangle_N}{N} = 1/2$$

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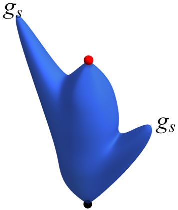

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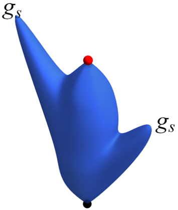
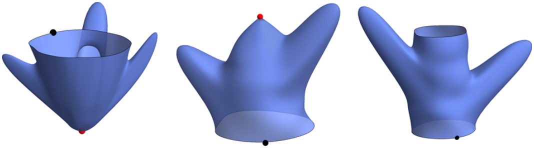
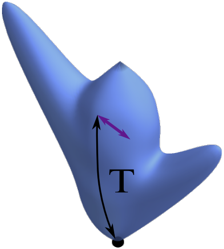


- Can compute:



Two-point functions

- Continuum amplitude for surfaces with root at distance T from origin.



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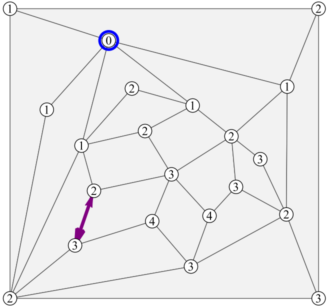

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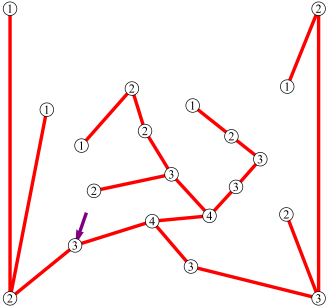

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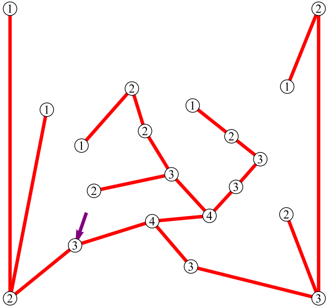
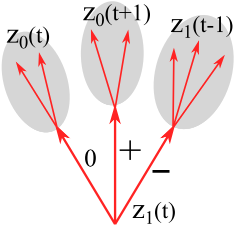
- Idem for $z_1(t)$, but local max at root not weighted by $\mathfrak{g}$. They satisfy



$$z_1(t) = \frac{1}{1 - gz_1(t-1) - gz_0(t) - gz_0(t+1)}$$

$$z_0(t) = z_1(t) + \frac{\mathfrak{g} - 1}{1 - gz_1(t-1) - gz_0(t)}$$

$$z_1(0) = 0 \quad z_0(\infty) = z_0$$



- Solution is (using methods of [Bouttier, Di Francesco, Guitter, '03]):

$$z_1(t) = z_1 \frac{1-\sigma^t}{1-\sigma^{t+1}} \frac{1-(1-\beta)\sigma-\beta\sigma^{t+3}}{1-(1-\beta)\sigma-\beta\sigma^{t+2}},$$

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with $\beta = \beta(g, \mathfrak{g})$ and $\sigma = \sigma(g, \mathfrak{g})$ fixed by

$$g(1+\sigma)(1+\beta\sigma)z_1 - \sigma(1-2g\, z_0) = 0,$$

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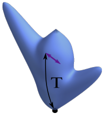
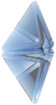


$$\sim \frac{dZ_0(T)}{dT} = \Sigma^3 \frac{\mathfrak{g}_s}{\alpha} \frac{\Sigma \sinh \Sigma T + \alpha \cosh \Sigma T}{\left(\Sigma \cosh \Sigma T + \alpha \sinh \Sigma T\right)^3}$$



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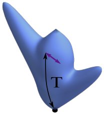

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$$\xrightarrow{\mathfrak{g}_s\to\infty} \Lambda^{3/4} \frac{\cosh(\Lambda^{1/4}T')}{\sinh^3(\Lambda^{1/4}T')} \qquad T' = \mathfrak{g}_s^{1/6}T$$

- DT two-point function appears as $\mathfrak{g}_s \to \infty$! [Ambjørn, Watabiki, '95]

Theorem 1^-

The map $\Psi_- : \mathcal{Q}^{(l)} \to \mathcal{M}^{(l)}$ is a bijection satisfying

- $\#\{\text{faces of } Q\} = \#\{\text{edges of } \Psi(Q)\}$
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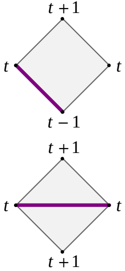
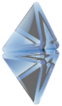
- $\#\{\text{local minima of } Q\} = \#\{\text{local minima of } \Psi(Q)\}$
- Min label on root edge of Q = label on root of $\Psi(Q)$
- Minimal label on Q = minimal label on $\Psi(Q)$





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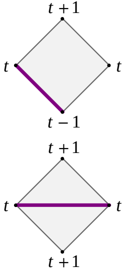

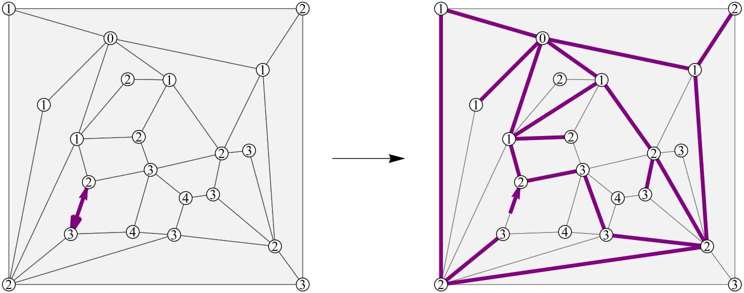

- Restricting Ψ^- to Q with single minimum labeled 0

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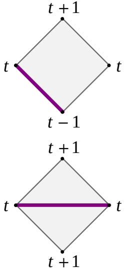
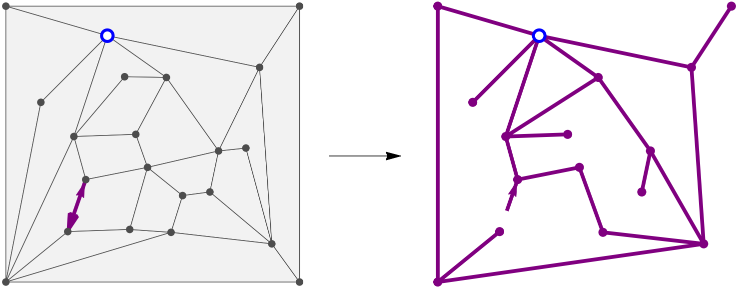
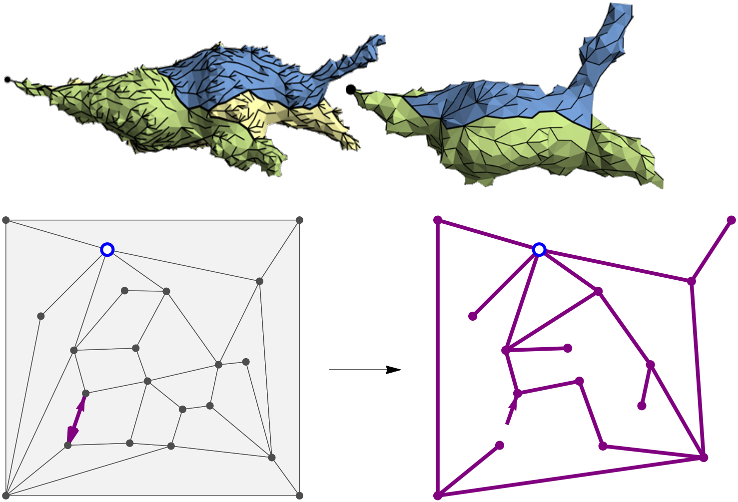




- Restricting Ψ^- to Q with single minimum labeled 0 gives a bijection between pointed rooted quadrangulations and pointed rooted planar maps.

- $\Psi^{-}(Q)$ has a face for each local maximum of Q. Within each face the structure of Q is similar to CDT.



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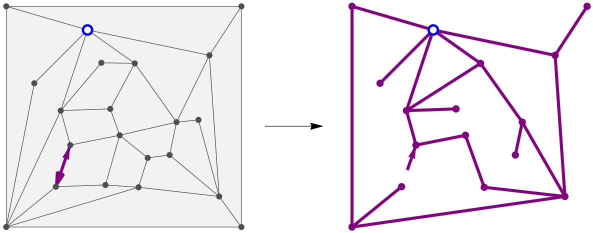
Two-point function for planar maps



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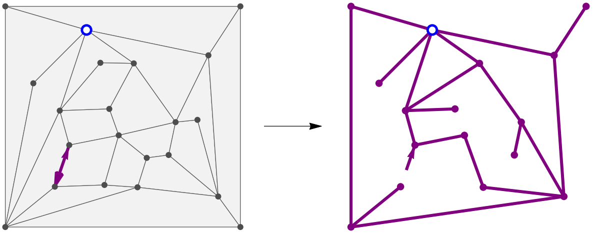


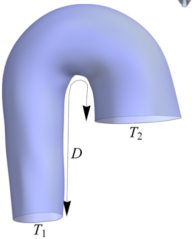
- The generating function for Q with max label t on the root edge is $z_0(t) - z_0(t-1)$
- Therefore one obtains an explicit generating function

$$z_0(t+1) - z_0(t) = \sum_{N=0}^{\infty} \sum_{n=0}^{\infty} \mathcal{N}_t(N, n) g^N \mathfrak{g}^n$$

for the number $\mathcal{N}_t(N, n)$ of planar maps with N edges, n faces, and a marked point at distance t from the root.

Two loop identity in generalized CDT

- Consider surfaces with two boundaries separated by a geodesic distance D.



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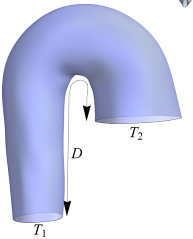
- One can assign time T_1, T_2 to the boundaries ($|T_1 - T_2| \leq D$) and study a "merging" process.



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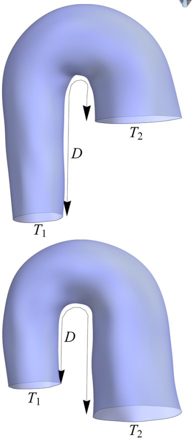
- One can assign time T_1, T_2 to the boundaries ($|T_1 - T_2| \leq D$) and study a "merging" process.
- For a given surface the foliation depends on $T_1 - T_2$, hence also N_{max} and its weight.



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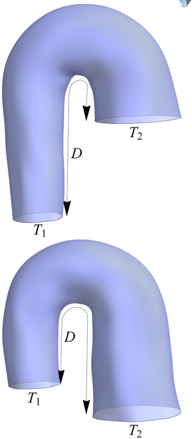

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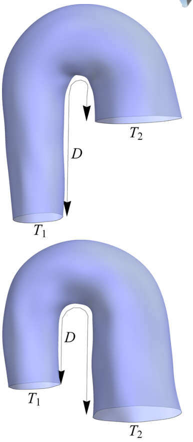

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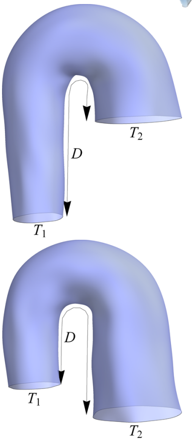

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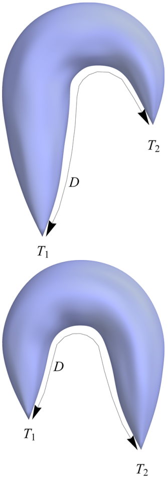
- For simplicity set the boundary lengths to zero. Straightforward generalization to finite boundaries.



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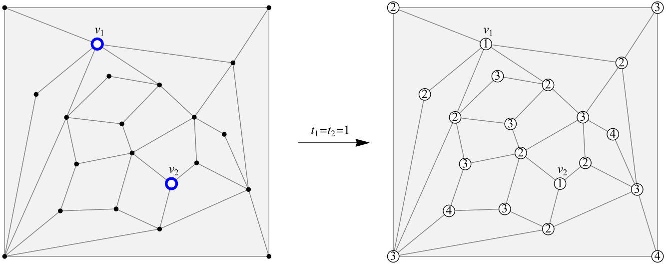

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$\nearrow$

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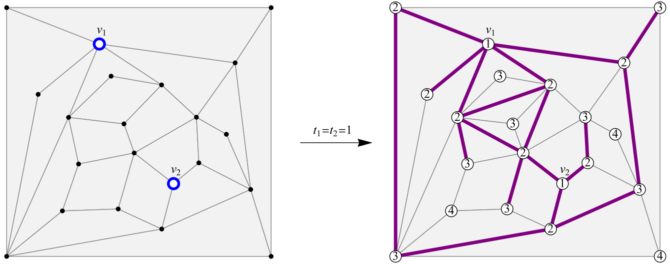

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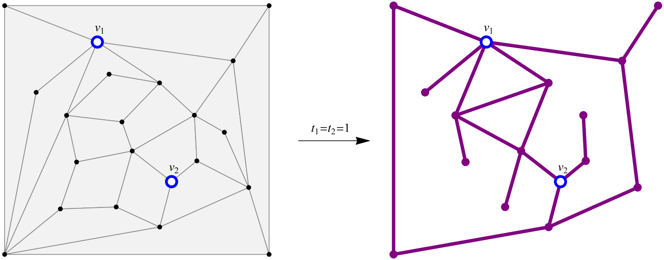

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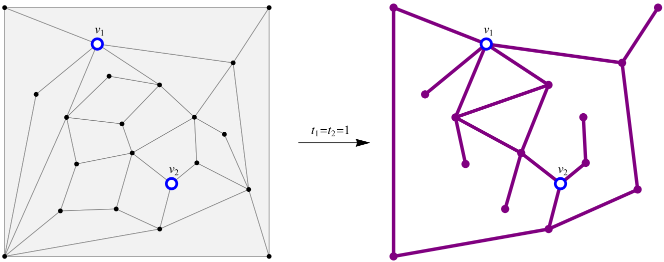

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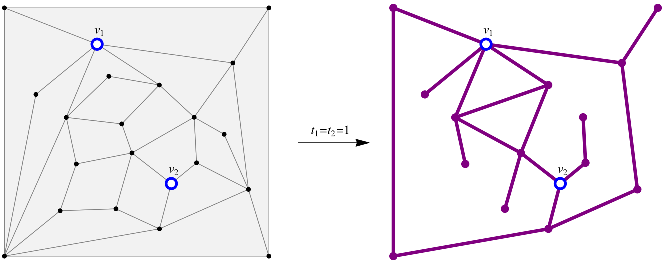

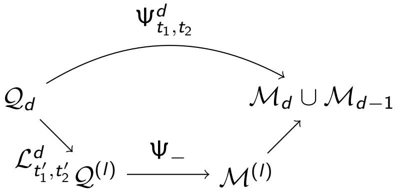

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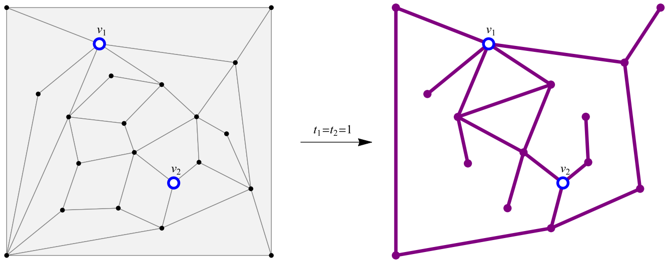

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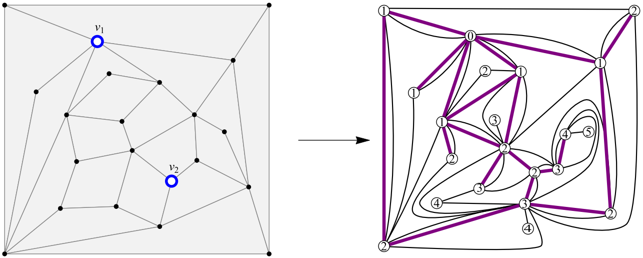

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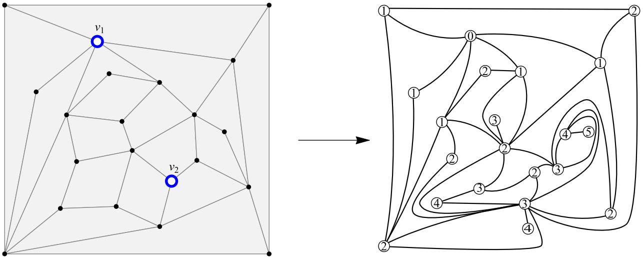

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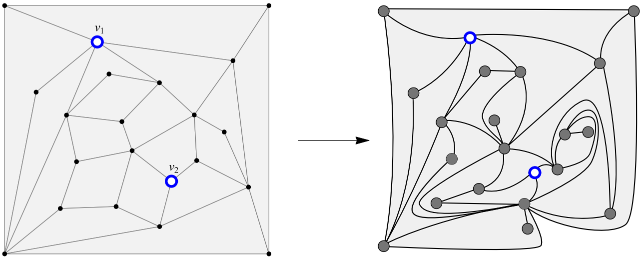

- We have found a bijection $(\Psi^d_{t_1',t_2'})^{-1} \circ \Psi^d_{t_1,t_2} : \mathcal{Q}_d \to \mathcal{Q}_d$ preserving the distance between v_1 and v_2, mapping $N_{\max}$ maxima w.r.t. (t_1, t_2) to $N_{\max}$ maxima w.r.t. (t_1', t_2').

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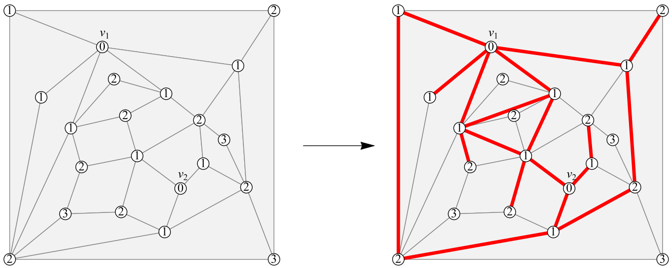

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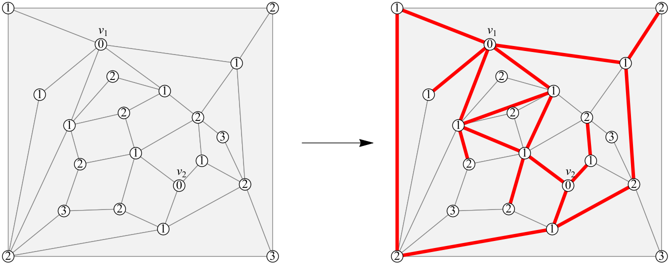

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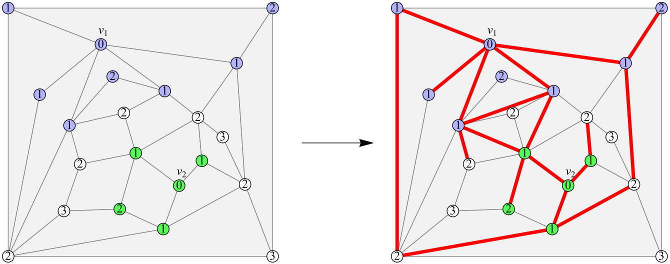

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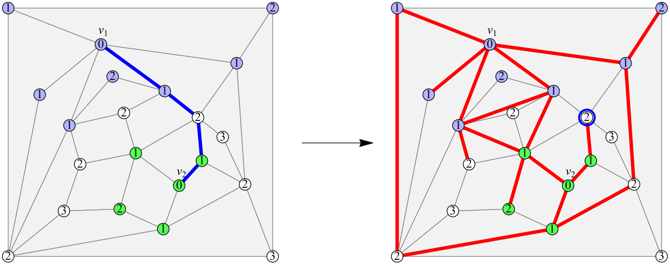

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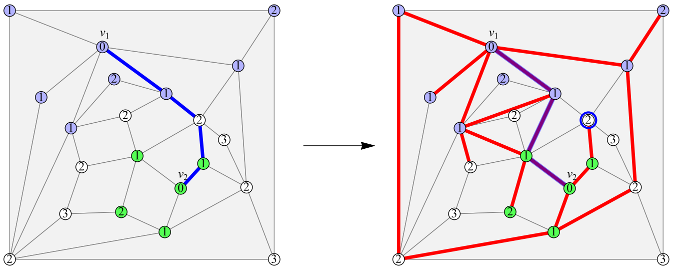

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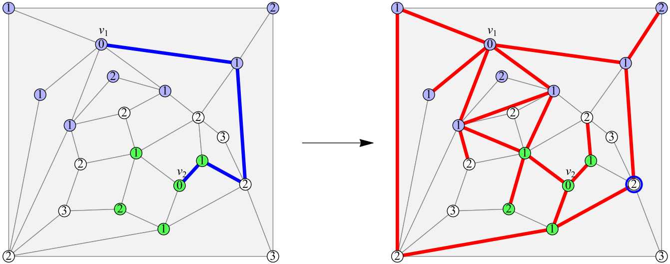

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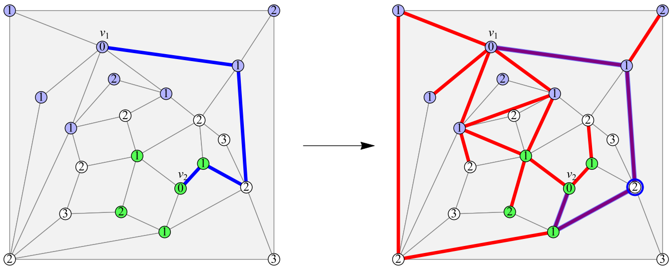

Conclusions & Outlook

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Further reading: arXiv:1302.1763

These slides and more: `http://www.nbi.dk/~budd/`

Thanks for your attention!

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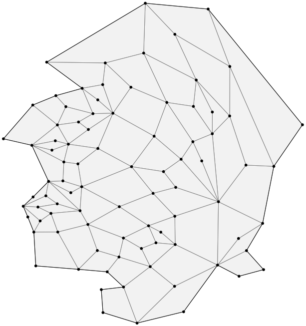
- A quadrangulations with boundary length $2l$



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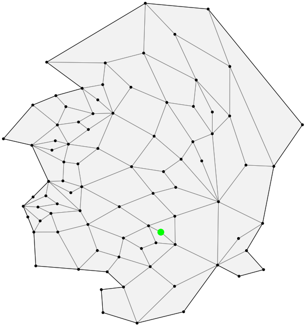

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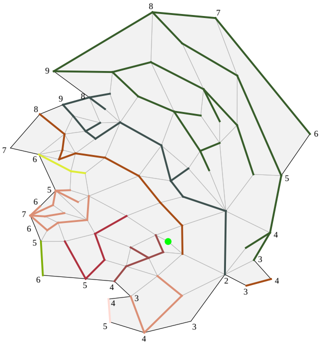
- Applying the same prescription we obtain a forest rooted at the boundary.



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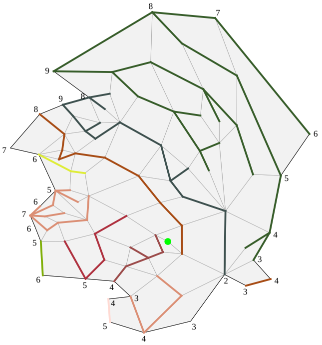

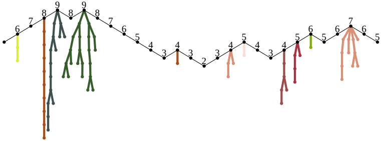

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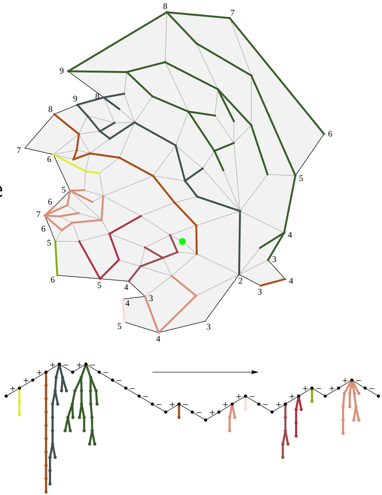

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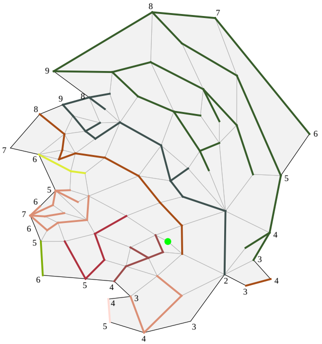
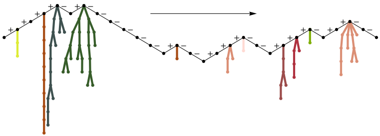
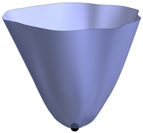
- There is a bijection

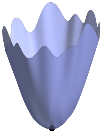




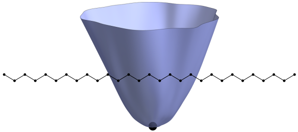
$$\left\{\begin{array}{l}\text{Quadrangulations with origin}\\ \text{and boundary length } 2l\end{array}\right\} \leftrightarrow \{(+,-)\text{-sequences}\} \times \{\text{tree}\}^l$$

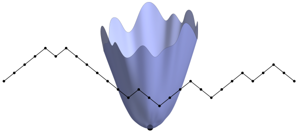
Disk amplitudes



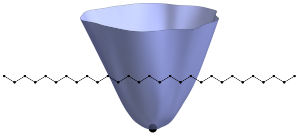


Disk amplitudes



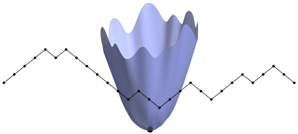


Disk amplitudes



$$w(g,l) = z(g)^l$$

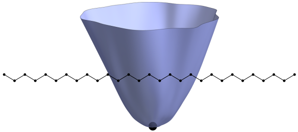
$$w(g,x) = \sum_{l=0}^{\infty} w(g,l)x^l = \frac{1}{1 - z(g)x}$$



$$w(g,l) = \binom{2l}{l} z(g)^l$$

$$w(g,x) = \frac{1}{\sqrt{1 - 4z(g)x}}$$

Disk amplitudes



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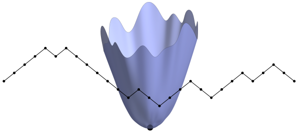

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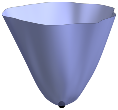
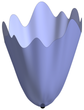
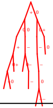
Generating function for unlabeled trees:

$$z(g) = \frac{1 - \sqrt{1 - 4g}}{2g}$$

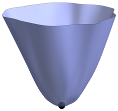
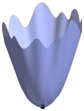
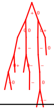
Generating function for labeled trees:

$$z(g) = \frac{1 - \sqrt{1 - 12g}}{6g}$$

Continuum limit

 $w(g,x) = \dfrac{1}{1 - zx}$	 $w(g,x) = \dfrac{1}{\sqrt{1 - 4zx}}$
 $z(g) = \frac{1-\sqrt{1-4g}}{2g}$	 $z(g) = \frac{1-\sqrt{1-12g}}{6g}$

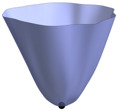
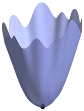
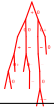
Continuum limit

 $w(g,x) = \dfrac{1}{1 - zx}$	 $w(g,x) = \dfrac{1}{\sqrt{1 - 4zx}}$
 $z(g) = \frac{1-\sqrt{1-4g}}{2g}$	 $z(g) = \frac{1-\sqrt{1-12g}}{6g}$

- Expanding around critical point in terms of "lattice spacing" ϵ:

$$g = g_c(1 - \Lambda\epsilon^2), \quad z(g) = z_c(1 - Z\epsilon), \quad x = x_c(1 - X\epsilon)$$

Continuum limit

 $w(g,x) = \frac{1}{1 - zx}$ $W_\Lambda(X) = \frac{1}{X + Z}$	 $w(g,x) = \frac{1}{\sqrt{1 - 4zx}}$ $W'_\Lambda(X) = \frac{1}{\sqrt{X + Z}}$
 $z(g) = \frac{1 - \sqrt{1-4g}}{2g}$ $Z = \sqrt{\Lambda}$	 $z(g) = \frac{1 - \sqrt{1-12g}}{6g}$ $Z = \sqrt{\Lambda}$

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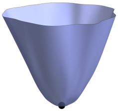
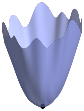
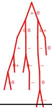
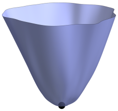
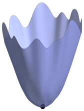
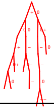
 $w(g,x) = \frac{1}{1-zx}$ $W_\Lambda(X) = \frac{1}{X+Z}$	 $w(g,x) = \frac{1}{\sqrt{1-4zx}}$ $W'_\Lambda(X) = \frac{1}{\sqrt{X+Z}}$
 $z(g) = \frac{1-\sqrt{1-4g}}{2g}$ $Z = \sqrt{\Lambda}$	 $z(g) = \frac{1-\sqrt{1-12g}}{6g}$ $Z = \sqrt{\Lambda}$

- Expanding around critical point in terms of "lattice spacing" ϵ:

$$g = g_c(1-\Lambda\epsilon^2), \quad z(g) = z_c(1-Z\epsilon), \quad x = x_c(1-X\epsilon)$$

- CDT disk amplitude: $W_\Lambda(X) = \frac{1}{X+\sqrt{\Lambda}}$

Continuum limit

 $w(g,x) = \frac{1}{1-zx}$ $W_\Lambda(X) = \frac{1}{X+Z}$	 $w(g,x) = \frac{1}{\sqrt{1-4zx}}$ $W'_\Lambda(X) = \frac{1}{\sqrt{X+Z}}$
 $z(g) = \frac{1-\sqrt{1-4g}}{2g}$ $Z = \sqrt{\Lambda}$	 $z(g) = \frac{1-\sqrt{1-12g}}{6g}$ $Z = \sqrt{\Lambda}$

- Expanding around critical point in terms of "lattice spacing" ϵ:

$$g = g_c(1-\Lambda\epsilon^2), \quad z(g) = z_c(1-Z\epsilon), \quad x = x_c(1-X\epsilon)$$

- CDT disk amplitude: $W_\Lambda(X) = \frac{1}{X+\sqrt{\Lambda}}$
- DT disk amplitude with marked point: $W'_\Lambda(X) = \frac{1}{\sqrt{X+\sqrt{\Lambda}}}$. Integrate w.r.t. Λ to remove mark: $W_\Lambda(X) = \frac{2}{3}(X - \frac{1}{2}\sqrt{\Lambda})\sqrt{X+\sqrt{\Lambda}}$.