Conformal point processes & planar maps

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work in progress (to appear soon)

Discrete 2d gravity

Continuous 2d gravity

recursion relations
integrability

Topological gravity

Random matrices

QFT
CFT
integrability

combinatorics

Liouville theory
&
Conformal field theories

Planar maps
(diagrams)

conformal invariance
probabilities

KPZ relations

Cori-Vauquelin-
Schaeffer-....
bijections
combinatorics
probabilities

SLE, GFF and all that

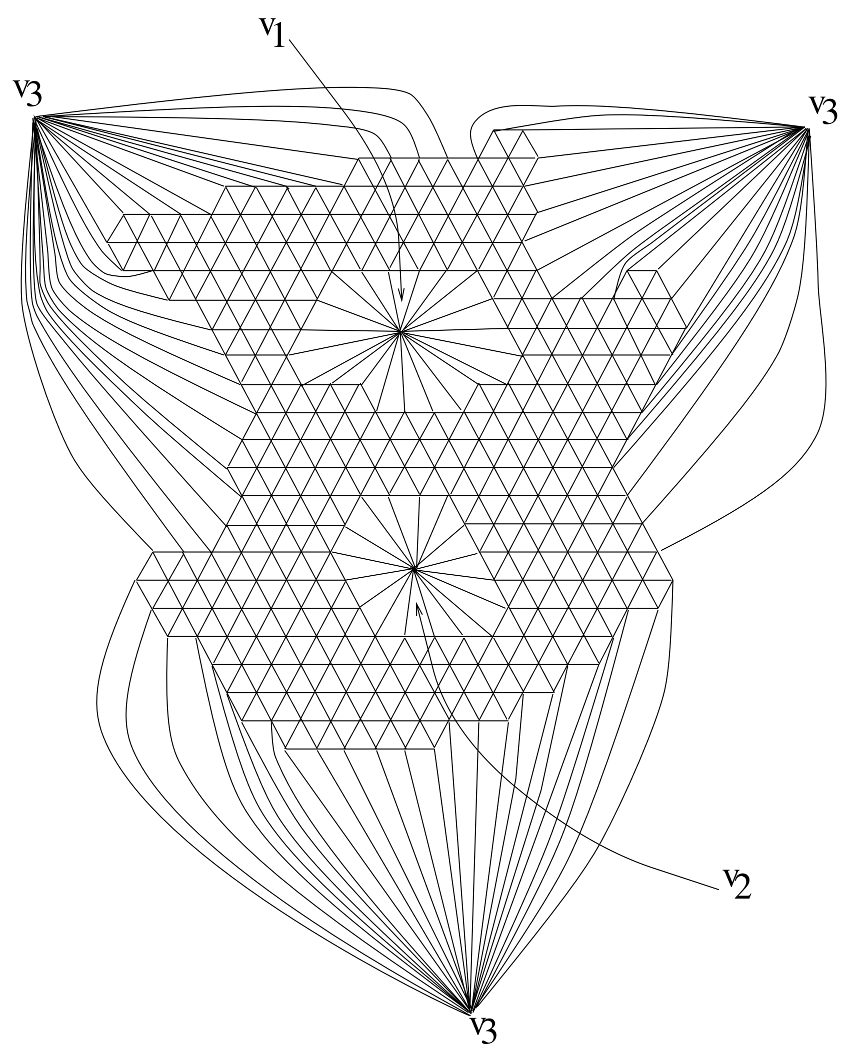
Brownian trees
and all that

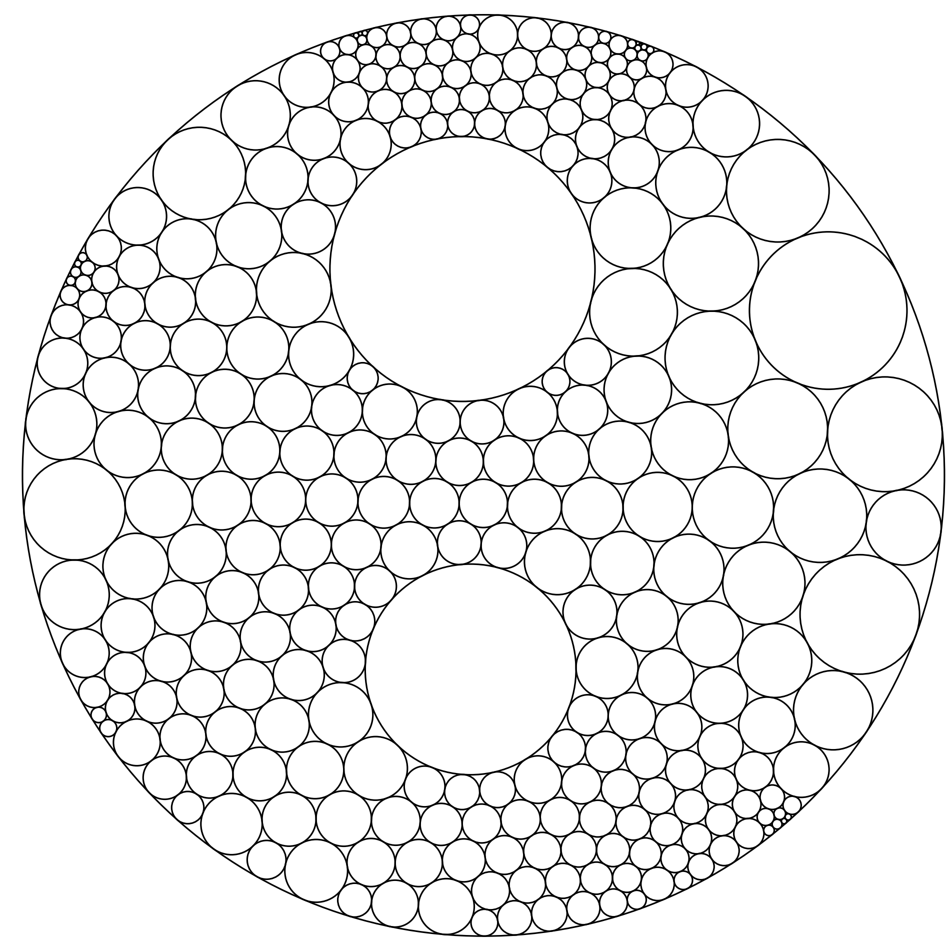
> Still lacking: a constructive link between the discrete and the continuous formulations

1. Introduction, circle packings and circle patterns
2. Delaunay circle patterns and (Euclidean) planar maps
3. Spanning 3-trees representation
4. Kähler geometry on Delaunay triangulation space
5. Discretized Faddev-Popov operator
6. (many) open or in progress questions

The Koebe-Andreev-Thurston theorem

crudely stated: there is a bijection between triangulations and circle packings, mod. SL(2,C)





Illustrations borrowed to Schramm & Mishenko

Circle packing are very useful to construct and study conformal and quasiconformal geometries

There is a deep relation between circle packings in 2d and hyperbolic geometry in 3d!

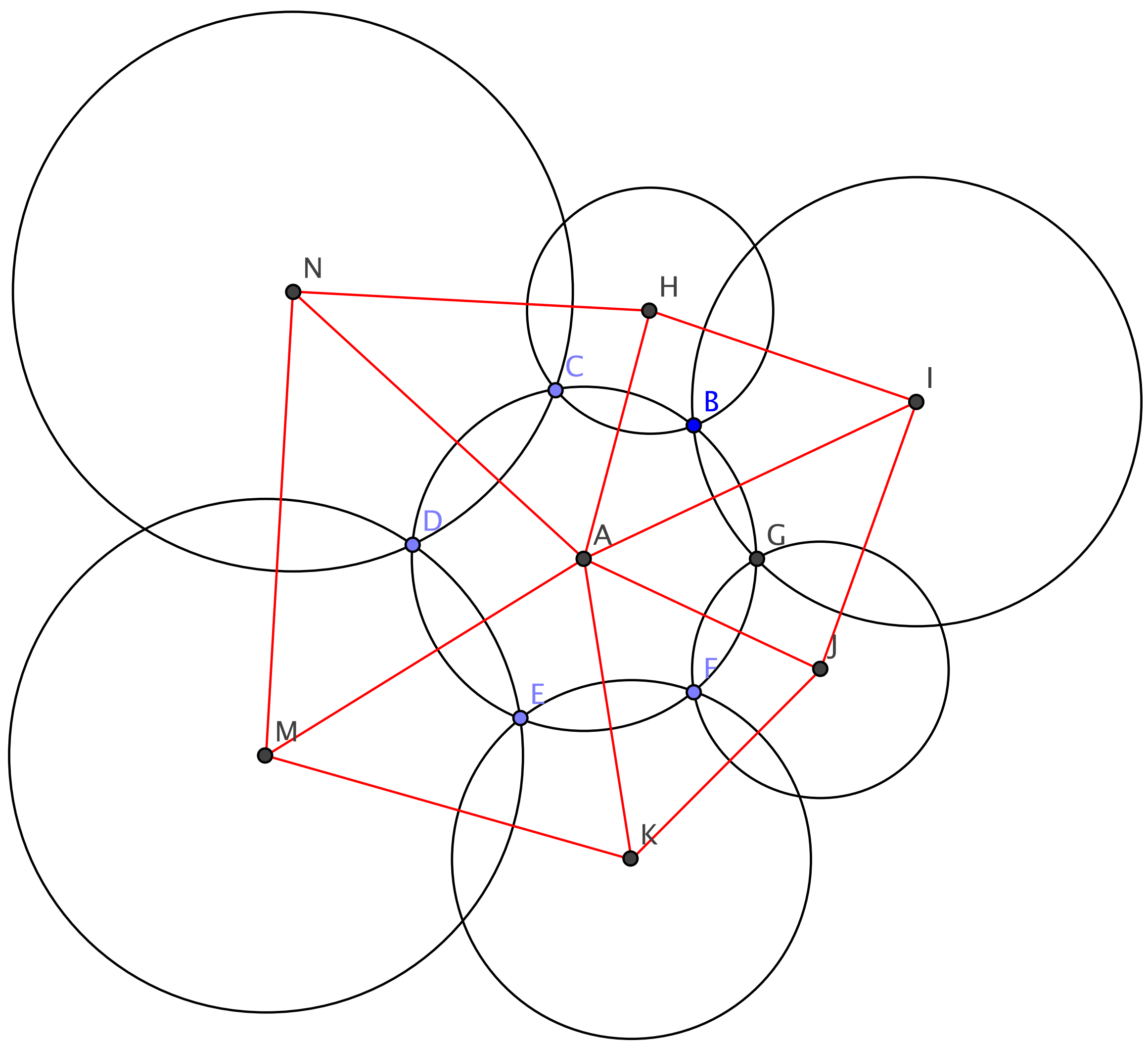
There is an idea «floating around» that they should be useful to make the link between discrete random geometry and continuous conformal geometry

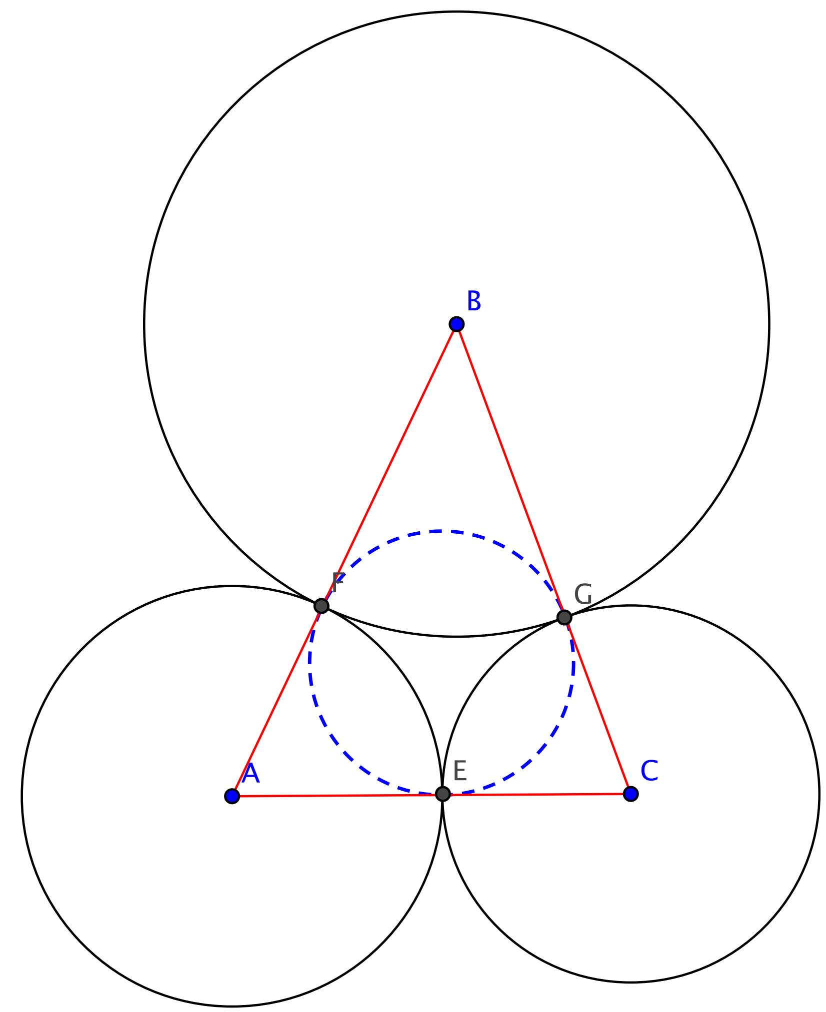
for instance to derive in a constructive way the relation for the Liouville background charge Q (KPZ 1988, F. David 1988)

$$Q^2 = \frac{25-c}{6} \qquad\qquad Q = \frac{2}{\sqrt{\kappa}} + \frac{\sqrt{\kappa}}{2}$$

A generalisation of circle packings: circle patterns

The angles of intersection of the circles are given.





Similar existence and unicity theorem (Rivin 1994, Ann. of Math.)

circle packing: angles 0 or $\pi/2$

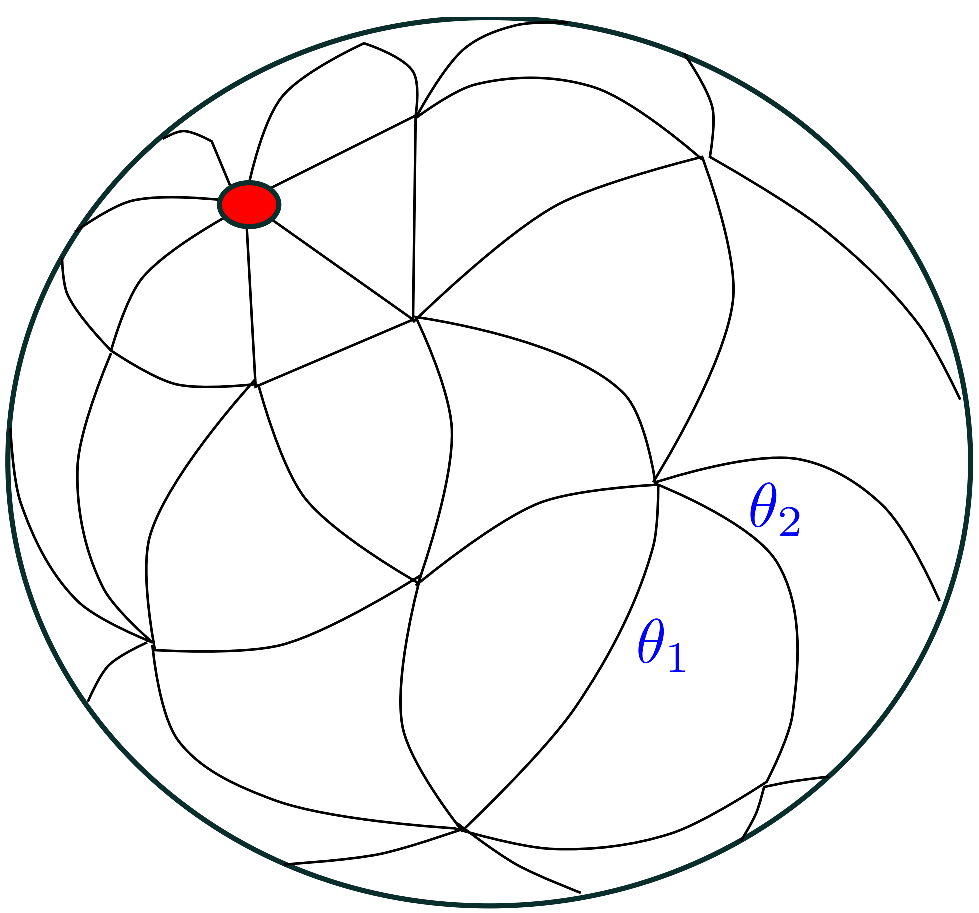
Application: Random Delaunay triangulations

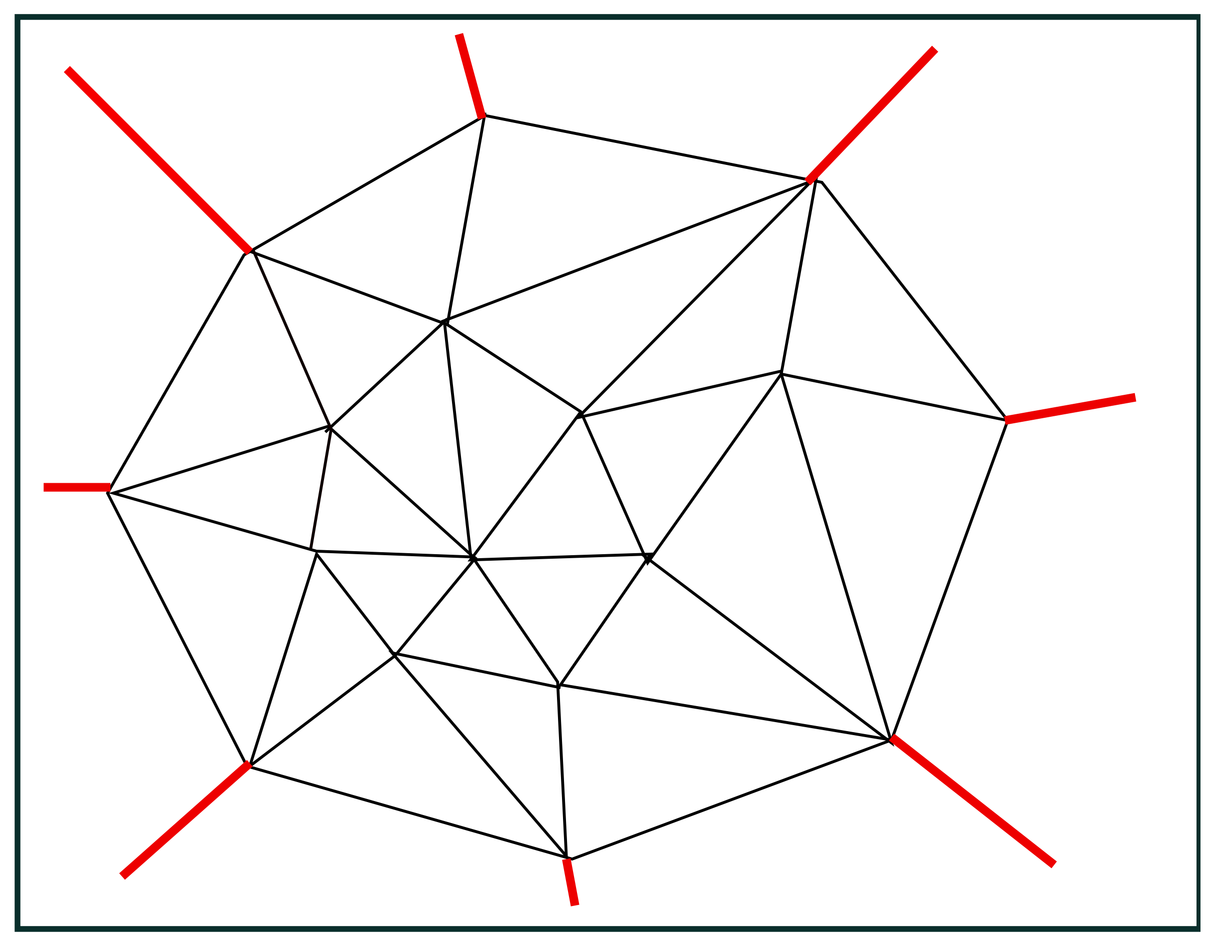
Consider an abstract triangulation of the sphere T

+ angles θ_e attached to the edges e of T

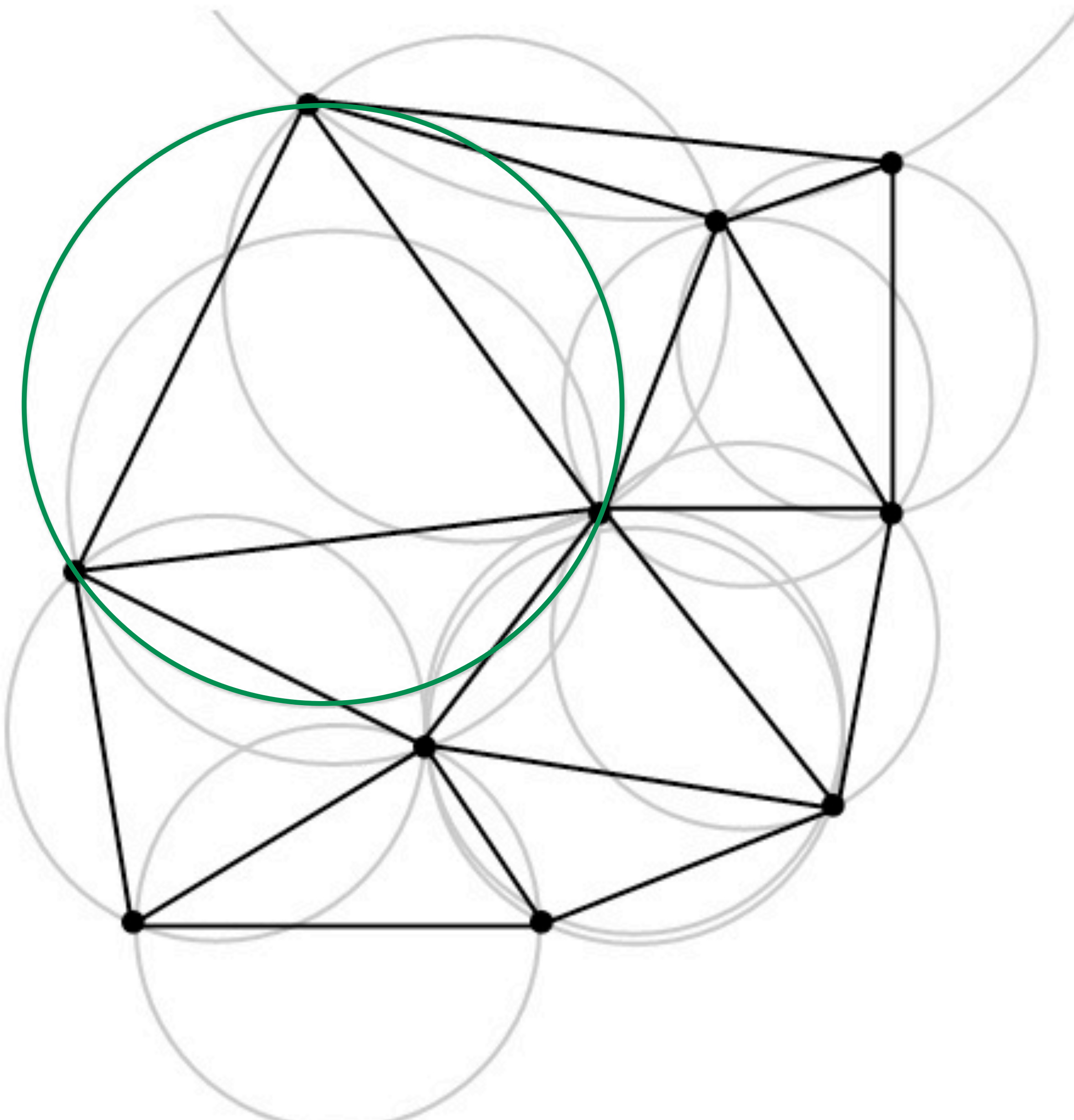
such that at each vertex $$\sum_{e \to v} \theta_e = 2\pi$$

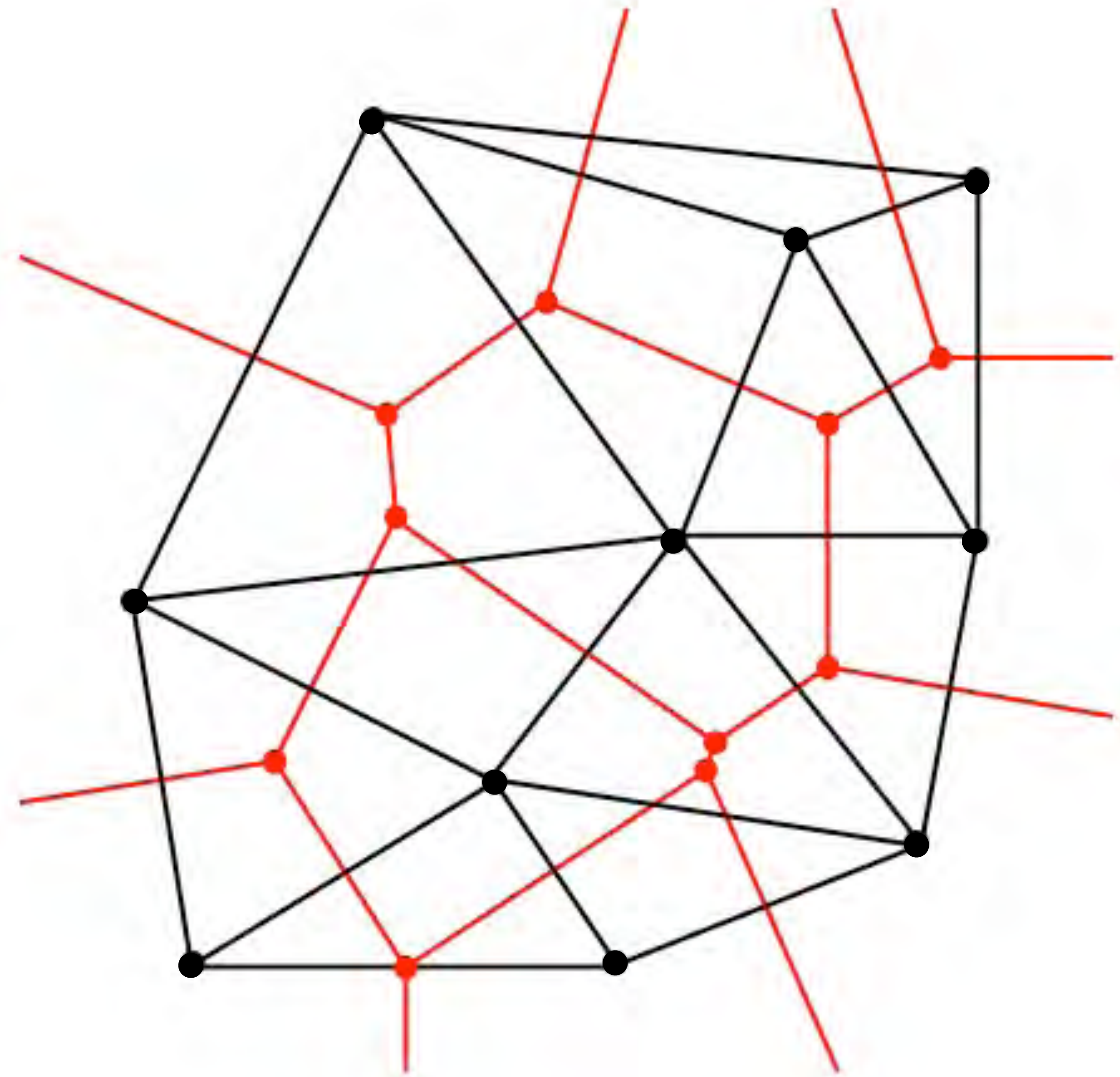
Theorem: there is a unique (up to SL(2,C)) Delaunay triangulation in the complex plane such that the circle angles are $\theta_e^* = \pi - \theta_e$





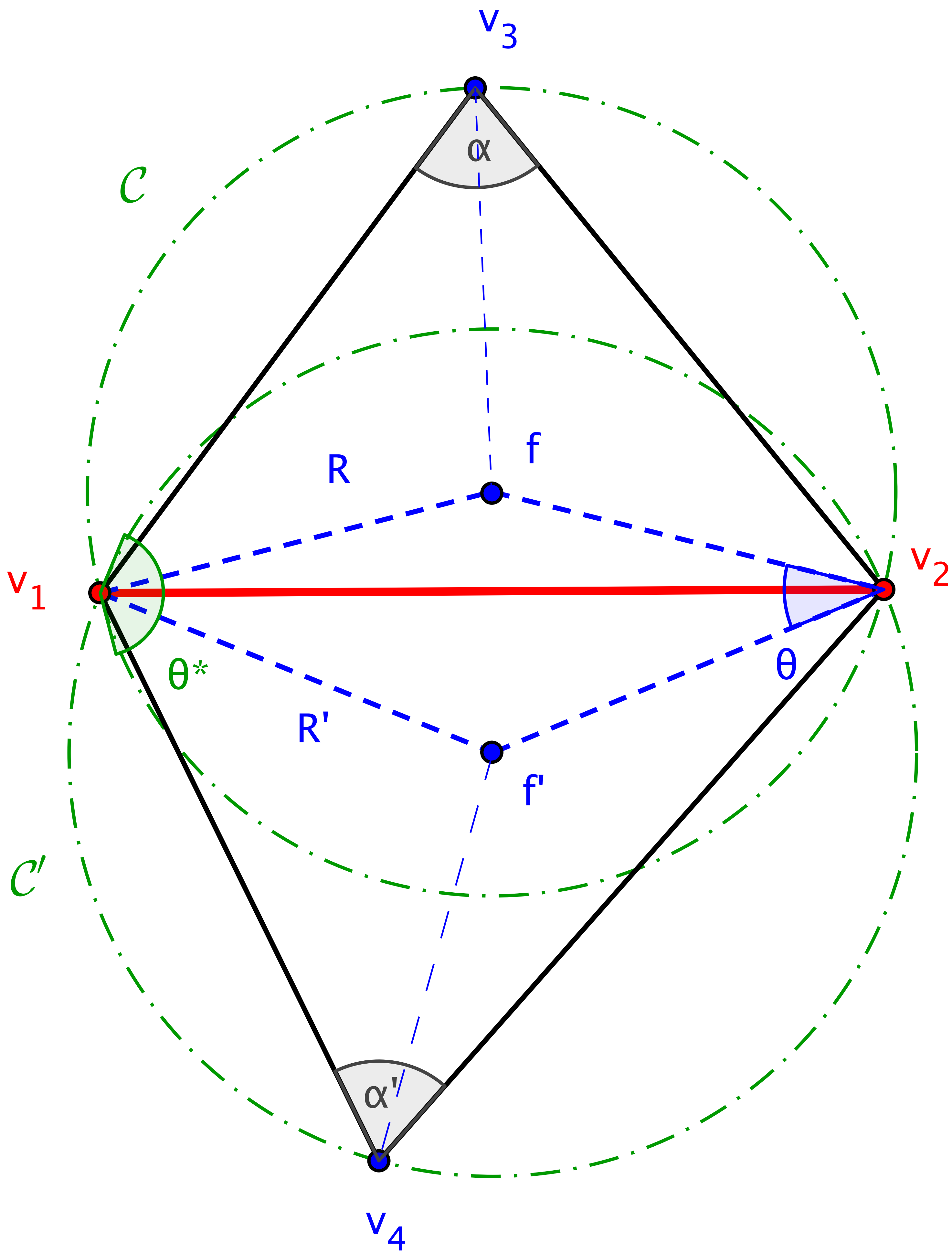
Reminder: Delaunay triangulation and Voronoï lattice





No vertex is inside the circumscribed circle to another triangle

From Wikimedia Commons


v3
α
C
f
R
v2
v1
θ*
θ
R'
f'
C'
α'
v4

NB: A triangulation is Delaunay iff all $\theta_e \in [0,\pi)$

Warning! Not all planar triangulations T admit $\boldsymbol{\theta}=\{\theta_e\}$ such that $\sum_{e\to v}\theta_e = 2\pi$

For instance, triangulations with loops or multiple links are excluded

But one expects that admissible $(T,\boldsymbol{\theta})$ are generic and in the same universality class than generic planar triangulations and general planar maps (more later...).

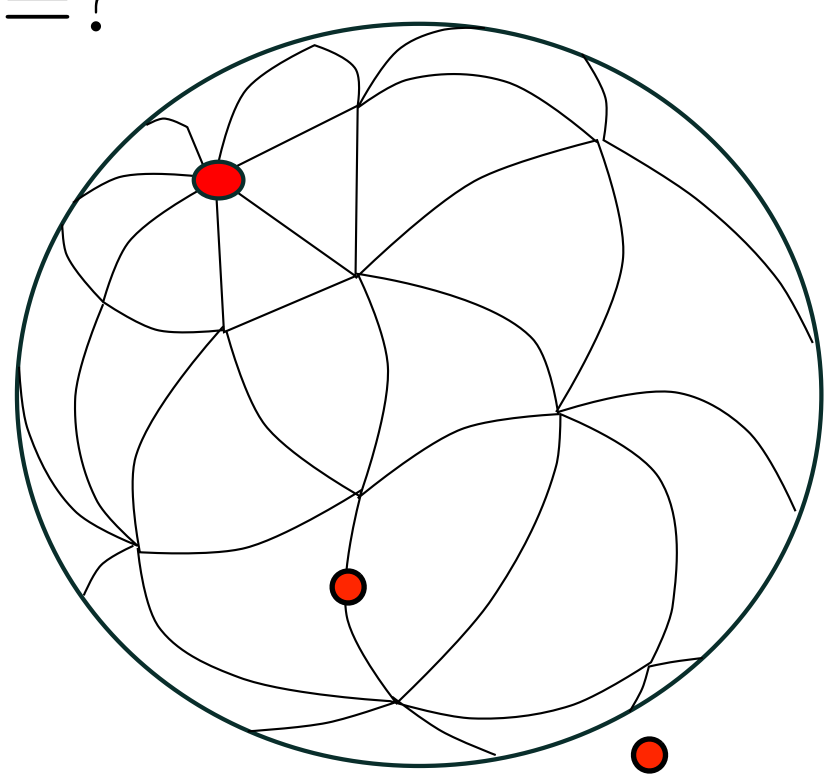
Take as initial measure on triangulations the uniform measure on triangulations and the flat measure on the angles

$$\mu(\widetilde{T}) = \mu(T, d\boldsymbol{\theta})) = \mathrm{uniform}(T) \prod_{e\in\mathcal{E}(T)} d\theta(e) \prod_{v\in\mathcal{V}(T)} \delta\left(\sum_{e\mapsto v} \theta(e) - 2\pi\right)$$

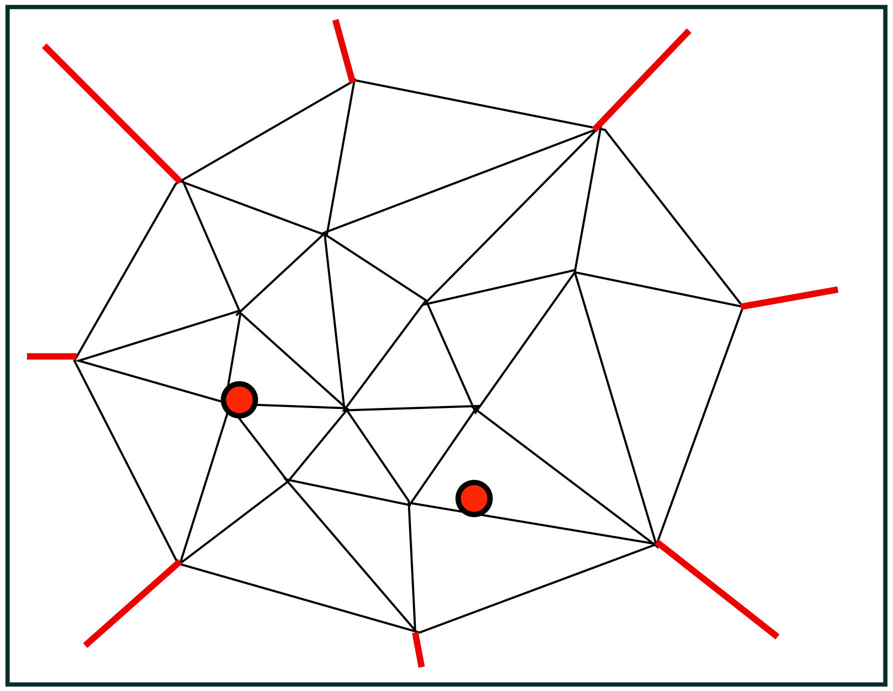
Question: which measure does this induce on Delaunay triangulations? For this consider $\mathrm{N+3}$ points, 3 fixed by $\mathrm{SL}(2,\mathbb{C})$

$$\mathfrak{D}_{N+3} = \mathbb{C}^{N+3}/\mathrm{SL}(2,\mathbb{C}) \simeq \mathbb{C}^N$$

$$d\mu(z_4, \cdots, z_{N+3}) = ?$$

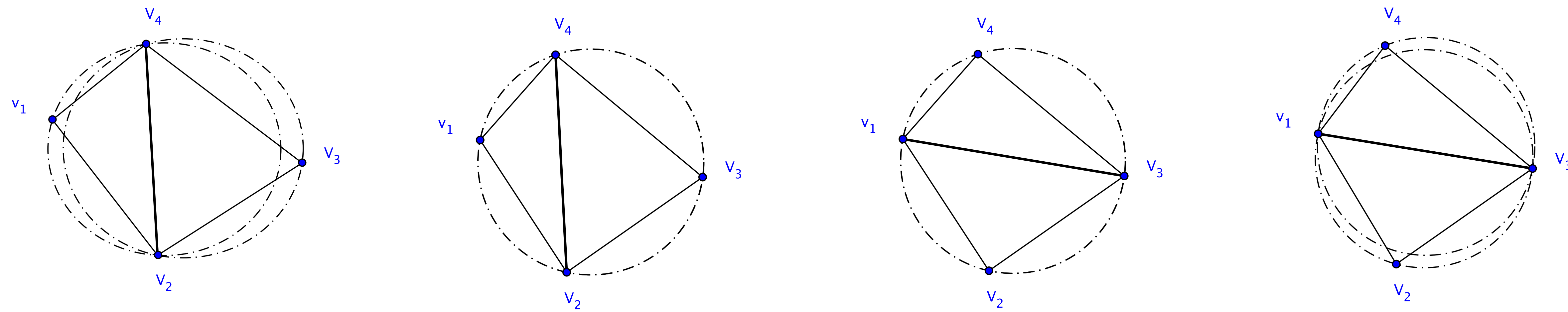


Sphere



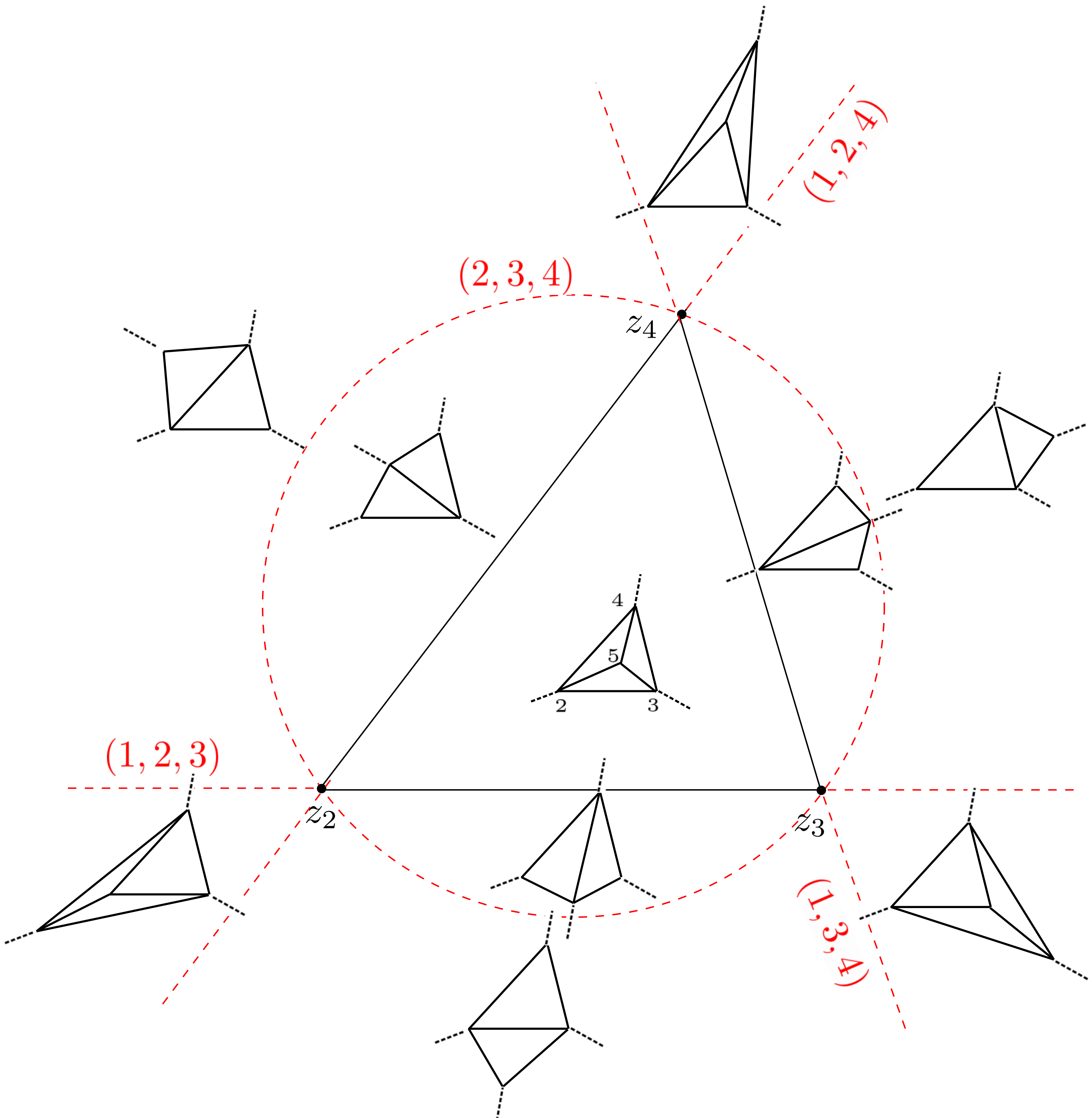
Plane

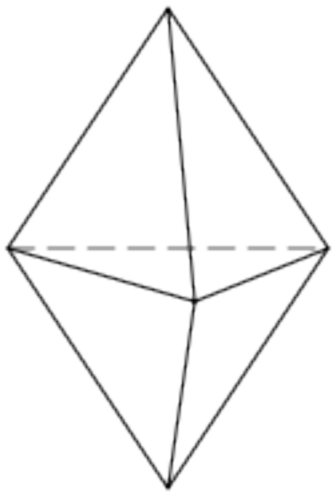
Transition between Delaunay triangulations by edge flips



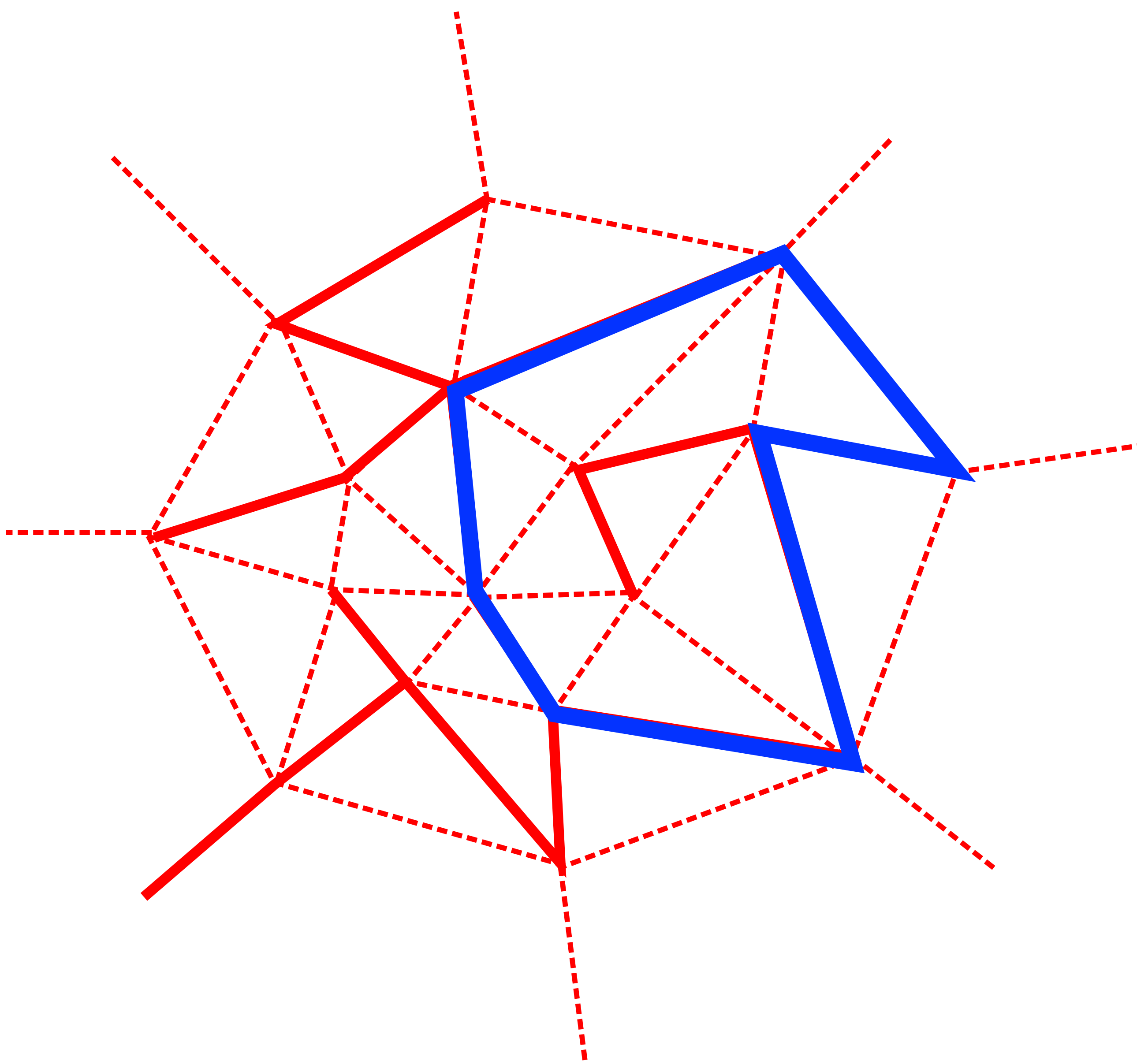
This occurs when $\theta(e) = 0$

Example: the hexahedron





1st question: Which sets of edges form independent basis for the angles?
Answer: Sets whose complementary form a cycle rooted spanning tree of the triangulation with odd length cycle



2nd question: what is the Jacobian of the change of angle variables between two basis of edges?

Answer: Jacobian = 1 !

Indeed... $\mu(T, d\boldsymbol{\theta}) \;=\; \frac{1}{2}\,\mathrm{uniform}(T) \times \prod_{e \in \mathcal{E}_0(T)} d\theta(e)$

So, the measure over the points is given locally (for a given Delaunay triangulation) by a simple Jacobian

$$\mu(T, d\boldsymbol{\theta}) \;=\; d\mu(\mathbf{z}) \;=\; \prod_{v=4}^{N+3} d^2 z_v \;\left|\det\left(J_T(z)_{\smallsetminus\{1,2,3\}\times\bar{\mathcal{E}}_0}\right)\right|$$

$$J_T(z) = \left(\frac{\partial\theta_e}{\partial(z_v, \bar{z}_v)}\right)_{\substack{e \in \mathcal{E}(T) \\ v \in \mathcal{V}(T)}}$$

Jacobian matrix elements sums of simple poles

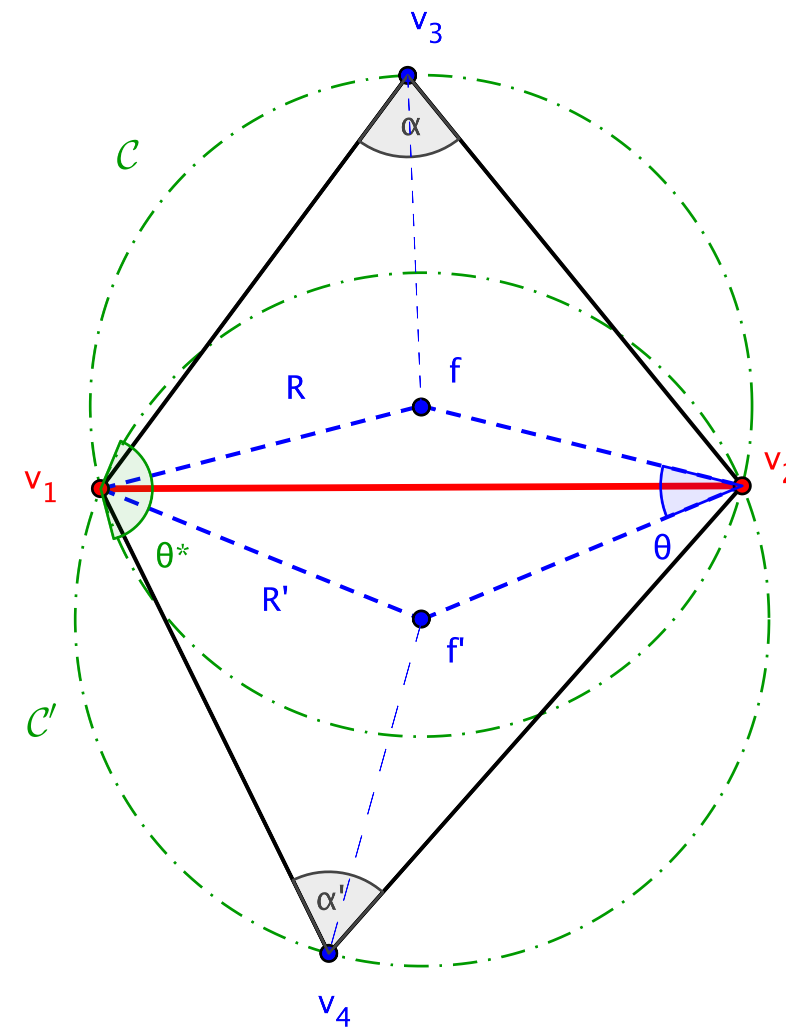
$$J_{v,e} = \frac{\partial \theta_e}{\partial z_v} \quad , \qquad J_{\bar{v},e} = \frac{\partial \theta_e}{\partial \bar{z}_v}$$

$$J_{v_1,e} = \frac{\mathrm{i}}{2}\left(\frac{1}{z_{v_4} - z_{v_1}} - \frac{1}{z_{v_3} - z_{v_1}}\right)$$

$$J_{v_3,e} = \frac{\mathrm{i}}{2}\left(\frac{1}{z_{v_3} - z_{v_1}} - \frac{1}{z_{v_3} - z_{v_2}}\right)$$

$$J_{v_2,e} = \frac{\mathrm{i}}{2}\left(\frac{1}{z_{v_3} - z_{v_2}} - \frac{1}{z_{v_4} - z_{v_2}}\right)$$

$$J_{v_4,e} = \frac{\mathrm{i}}{2}\left(\frac{1}{z_{v_4} - z_{v_2}} - \frac{1}{z_{v_4} - z_{v_1}}\right)$$



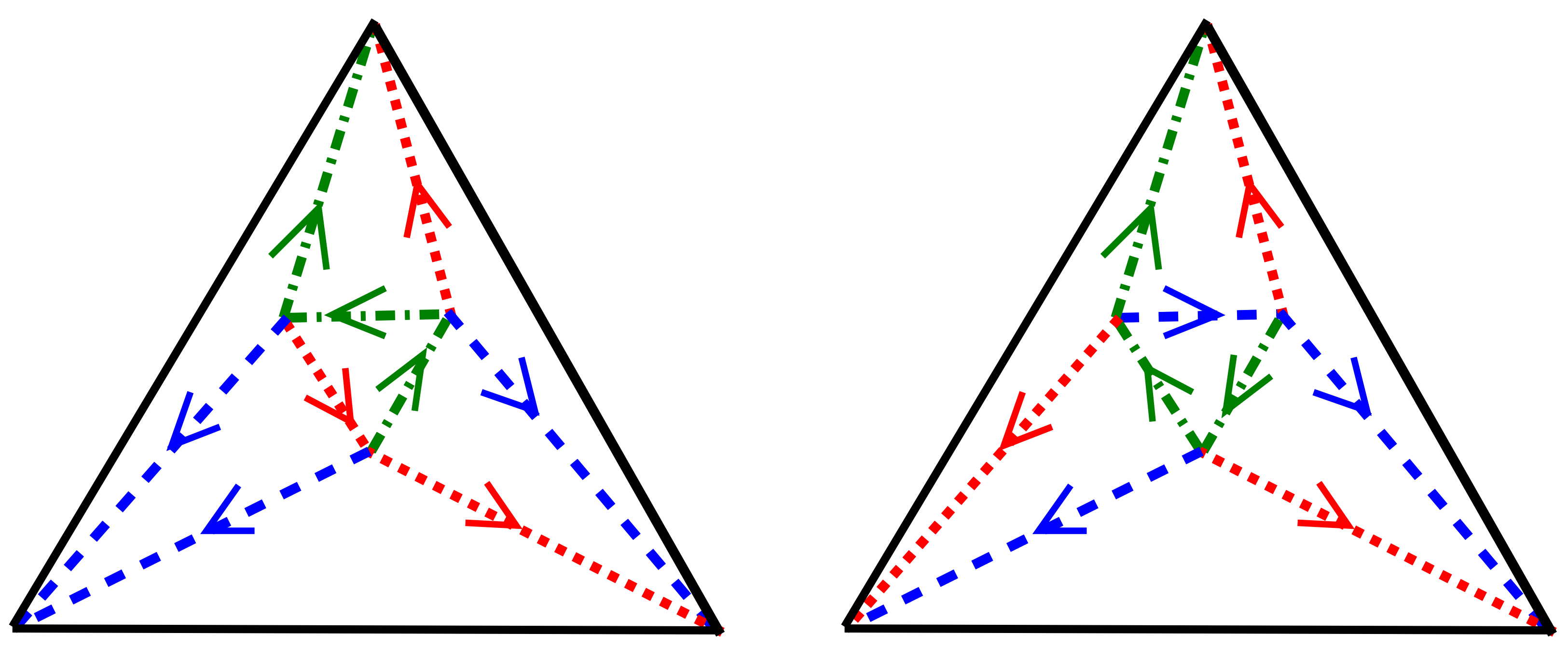
Jacobian determinant is locally a rational function

$$\mathcal{D}_T(z)_{\smallsetminus\{1,2,3\}} = \left|\det\left(J_T(z)_{\smallsetminus\{1,2,3\}\times\bar{\mathcal{E}}_0}\right)\right|$$

Definition 2.3 (triangle rooted spanning 3-tree)
Let T be a planar triangulation with $N+3$ vertices, and $\triangle = f_0$ be a face (triangle) of T, with 3 vertices $\mathcal{V}(\triangle) = (v_1, v_2, v_3)$ and 3 edges $\mathcal{E}(\triangle) = (e_1, e_2, e_3)$. Let $\mathcal{E}(T)_{\smallsetminus\triangle} = \mathcal{E}(T) \smallsetminus \mathcal{E}(\triangle)$ be the set of $3N$ edges of T not in $\triangle$.

We call a $\triangle$-rooted 3-tree of T ($\triangle R3T$) a family $\mathcal{F}$ of three disjoint *subsets $(\mathcal{I}, \mathcal{I}', \mathcal{I}'')$ of edges of $\mathcal{E}(T)$ such that:*

1. *$(\mathcal{I}, \mathcal{I}', \mathcal{I}'')$ are disjoint and disjoint of $\mathcal{E}(\triangle)$*
2. *Each $\mathcal{I} \cup \mathcal{E}(\triangle)$, $\mathcal{I}' \cup \mathcal{E}(\triangle)$, $\mathcal{I}'' \cup \mathcal{E}(\triangle)$ is a cycle rooted spanning tree of T with cycle $\triangle$.*



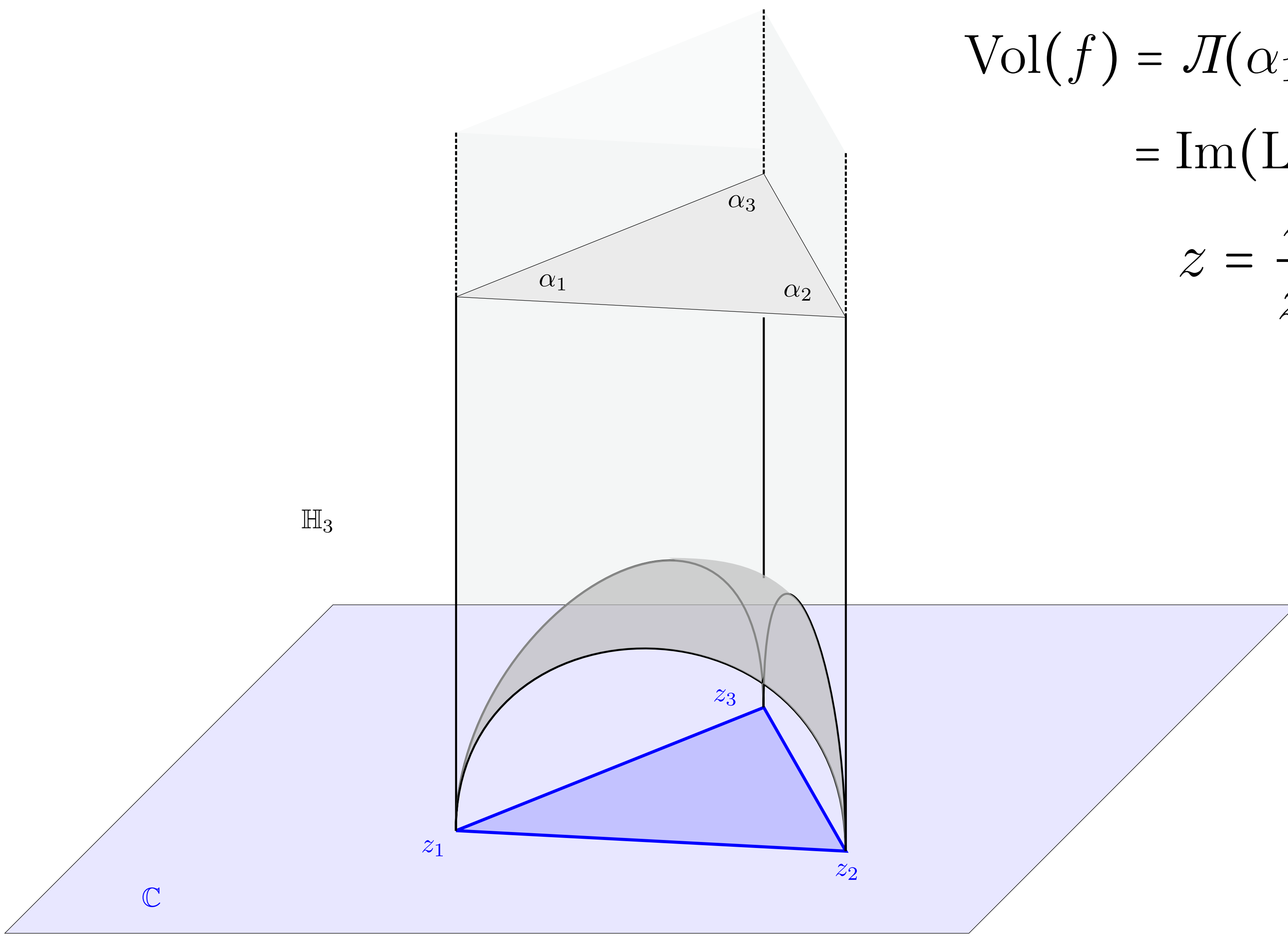
Theorem 2.3 *Let T be a planar triangulation of the plane with $N+3$ vertices. If the 3 fixed points (v_1, v_2, v_3) belong to a triangle (face) of T, the measure determinant takes the form*

$$\mathcal{D}_T(z)_{\smallsetminus\{1,2,3\})} = 4^{-N} \sum_{\substack{\mathcal{F}=(\mathcal{I},\mathcal{I}',\mathcal{I}'') \\ \triangle R3T \text{ of } T}} \epsilon(\mathcal{F}) \times \prod_{\vec{e}=(v\to v')\in\mathcal{I}} \frac{1}{z_v - z_{v'}} \times \prod_{\vec{e}=(v\to v')\in\mathcal{I}'} \frac{1}{\bar{z}_v - \bar{z}_{v'}} \qquad (2.10)$$

where $\epsilon(\mathcal{F}) = \pm 1$ is a sign factor, coming from the topology of T and of $\mathcal{F}$, that will be defined later.

This is a non trivial extension of the spanning tree representation of the determinant of scalar Laplacian(s). Specific to this matrix.

Hyperbolic volume of triangle = volume of ideal tetrahedron above the triangle in hyperbolic Poincaré half-space



$$\mathrm{Vol}(f) = Л(\alpha_1) + Л(\alpha_2) + Л(\alpha_3)$$

$$= \mathrm{Im}(\mathrm{Li}_2(z)) + \ln(|z|)\mathrm{Arg}\,(1-z)$$

$$z = \frac{z_3 - z_1}{z_2 - z1}$$

Action of a triangulation = sum over volumes

$$\mathcal{A}_T \;=\; - \sum_{triangles\;\; f \in \mathcal{F}(T)} \mathrm{Vol}(f)$$

Define $$D_{u\bar{v}}(z) = \frac{\partial}{\partial z_u}\frac{\partial}{\partial \bar{z}_v}\mathcal{A}_T(z)$$

Theorem:

$D_{u\bar{v}}$ is a Kähler form on $\mathfrak{D}_{N+3}$ i.e. $D > 0$

$D_{u\bar{v}}$ is countinuous (no discontinuity when a flip occurs)

The measure determinant is the Kähler volume form

$$\mathcal{D}_T(z)_{\setminus\{1,2,3\})} = \det\left[(D_{u,\bar{v}})_{\substack{u,v\neq \\ \{1,2,3\}}}\right]$$

The *(2N x 2N)* Jacobian has been reduced to a *N x N* Kähler determinant!

$d\mu(z) = d^2 z \ \det(D)$ is a conformal point process

Independence of the 3 fixed points and $\mathrm{SL}(2,\mathbb{C})$ invariance

$$H = \frac{\det\left(D_{\setminus a,b,c}(z)\right)}{\left|\Delta_3(z_a, z_b, z_c)\right|^2}$$

$$\Delta_3(z_a, z_b, z_c) = (z_a - z_b)(z_a - z_c)(z_b - z_c)$$

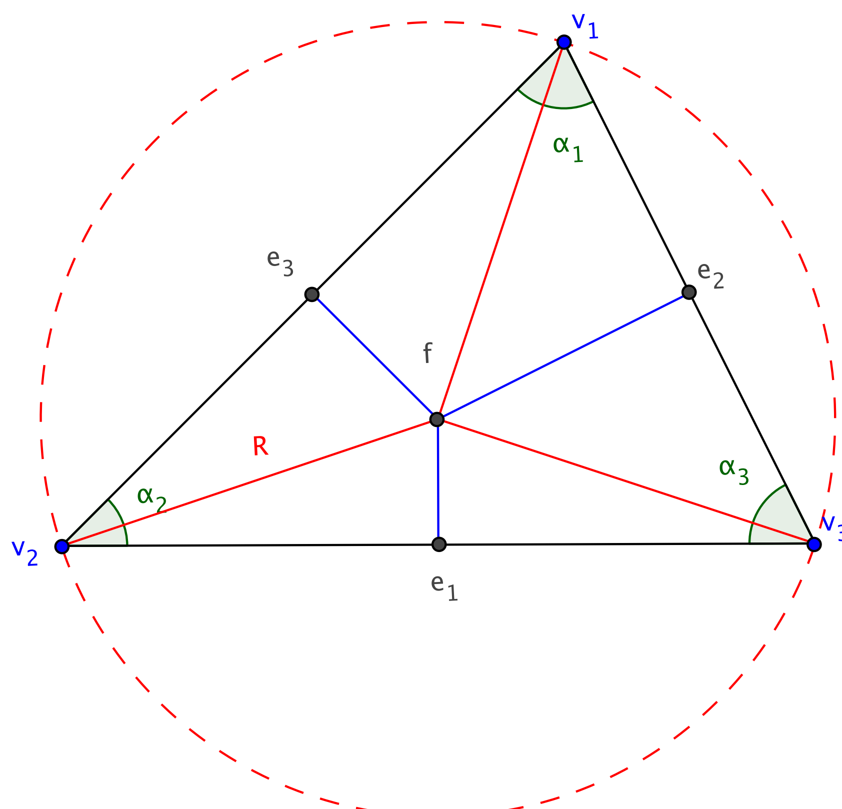
is independent of the choice of points

$$z \to w = \frac{az+b}{cz+d} \quad \text{with} \quad ad - bc = 1$$

$$H(z) = \left|\prod_{i=1}^{N+3} w'(z_i)\right|^2 H(w) = \prod_{i=1}^{N+3} \frac{1}{\left|cz_i + d\right|^2} H(w)$$

Local representation of D as a sum over triangles

$$D_{u,\bar v} = \sum_f D_{u,\bar v}(f) \quad , \qquad D_{u,\bar v}(f) \ = - \frac{\partial^2}{\partial z_u \partial \bar z_v} \mathrm{Vol}(f)$$



$$D(f) \ = \ \frac{1}{8\,R(f)^2} \begin{pmatrix} \cot(\alpha_2)+\cot(\alpha_3) & -\cot(\alpha_3)-\mathrm{i} & -\cot(\alpha_2)+\mathrm{i} \\ -\cot(\alpha_3)+\mathrm{i} & \cot(\alpha_3)+\cot(\alpha_1) & -\cot(\alpha_1)-\mathrm{i} \\ -\cot(\alpha_2)-\mathrm{i} & -\cot(\alpha_1)+\mathrm{i} & \cot(\alpha_1)+\cot(\alpha_2) \end{pmatrix}$$

D as a discretized Fadeev-Popov determinant!

Local derivatives from vertices $\longrightarrow$ faces

$$\nabla\Phi(f) = \frac{1}{2\mathrm{i}}\frac{\Phi(v_1)(\bar z_3-\bar z_2)+\Phi(v_2)(\bar z_1-\bar z_3)+\Phi(v_3)(\bar z_2-\bar z_1)}{\mathrm{Area}(f)}$$

$$\overline{\nabla}\Phi(f) = -\frac{1}{2\mathrm{i}}\frac{\Phi(v_1)(z_3-z_2)+\Phi(v_2)(z_1-z_3)+\Phi(v_3)(z_2-z_1)}{\mathrm{Area}(f)}$$

D can be written as

$$\Phi\cdot D(f)\cdot\overline{\Psi} = \sum_{i,j\ \text{vertices of}\ f}\Phi(v_i)D_{i\bar j}(f)\overline{\Psi}(v_j) = \frac{\mathrm{Area}(f)}{R(f)^2}\ \overline{\nabla}\Phi(f)\ \nabla\overline{\Psi}(f)$$

«continuous form» with complex functions treated as real vector fields

$$\mathrm{Area}(f) = d^2w_f \qquad \frac{1}{R(f)^2} = \mathrm{e}^{\,\phi(w_f)}$$

$$\Phi\cdot D\cdot\overline{\Psi} = \int d^2w\ \mathrm{e}^{\,\phi(w)}\ \partial_{\bar z}\Phi^z(w)\ \partial_z\Psi^{\bar z}(w)$$

This similar to the Faddeev-Popov determinant in Polyakov formulation of 2d gravity and string theory!

Functional integral over 2d Riemannian metrics, conformal gauge

$$g_{ab}(z) = \delta_{ab}\ \mathrm{e}^{\,\phi(z)} \qquad\qquad \int \mathcal{D}[g_{ab}]\ =\ \int \mathcal{D}[\phi]\ \det(\nabla_{\mathrm{FP}})$$

Faddeev-Popov ghost systems

$$\det(\nabla_{\mathrm{FP}})\ =\ \int \mathcal{D}[\mathbf{c},\mathbf{b}]\ \exp\left(\int d^2z\ \mathrm{e}^{\,\phi}\ \left(b_{zz}(\nabla c)^{zz} + b_{\bar z\bar z}(\nabla c)^{\bar z\bar z}\right)\right)$$

Integrating over the b's only one gets

$$\det(\nabla_{\mathrm{FP}})\ =\ \int \mathcal{D}[\mathbf{c}]\ \exp\left(\int d^2z\ \mathrm{e}^{\,\phi}\,\partial_z c^{\bar z}\ \partial_{\bar z} c^{z}\right)$$

D is nothing but the discretised FP determinant

$$D\ =\ \nabla_{\mathrm{FP}}$$

and $\phi(f) = -\,2\,\log(R(f))$

plays the role of a Liouville field on the Voronoï lattice

Many open questions and work in progress

Conformal properties of the measure
Is it possible to define a conformal stress-energy tensor?
Is this model integrable?
Is there an explicit local expression for $\mathrm{Tr}(\mathrm{Log}(D))$?
What is the central charge? How to compute it?
Continuum limit?
Is Liouville theory an effective large distance theory or is it already present in this formulation?
Can this approach be generalised to other quasi-conformal mappings or fully conformal mappings of random triangulations?
Coupling to statistical models and to matter?
Surfaces with boundaries, non-planar topologies?
etc...

Thank you!