Domain Markov half planar maps

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Joint work with O. Angel.

June 20, 2013

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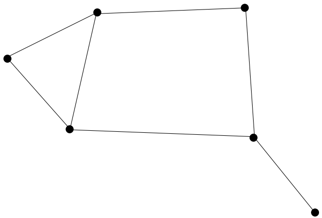

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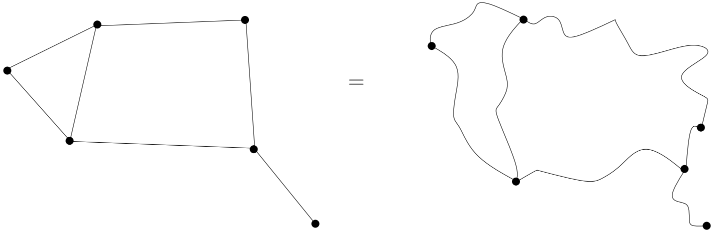

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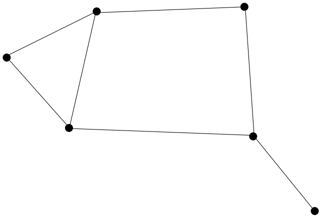

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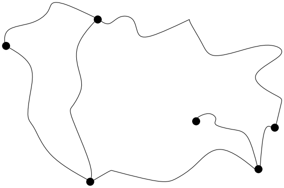


Local Metric

The topology (Local Metric):
Let B_r be the ball of radius r (in graph distance) around the root.
For any two planar maps T, T'

$$d(T, T') = (1 + sup\{r : B_r(T) = B_r(T')\})^{-1}$$

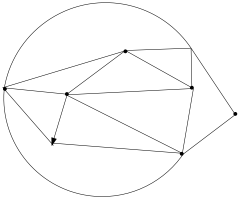
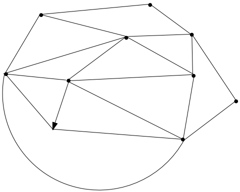
 

Figure : Here $d(T, T') = 1/2$

Background and motivation

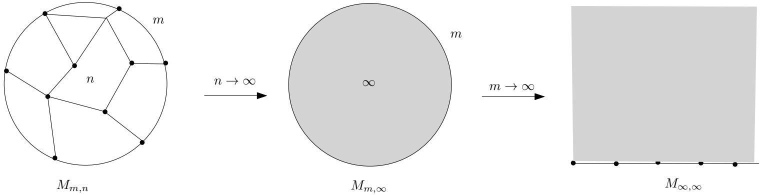
Let μ_n be the uniform measure on triangulations of the sphere.

Theorem (Angel,Schramm; 2002)

Weak limits of μ_n in local topology exists.

Similar result by Krikun, 2006 for quadrangulation.

Limits: *Uniform infinite planar triangulation/quadrangulation*(UIPT/Q)



We get: Half planar UIPT/Q.

Half planar Maps

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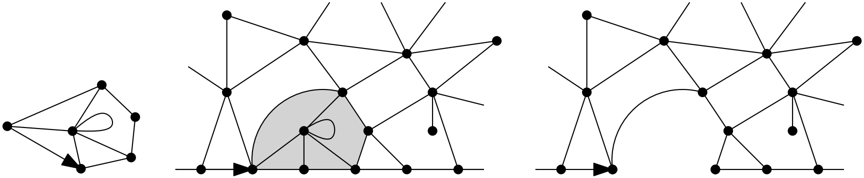
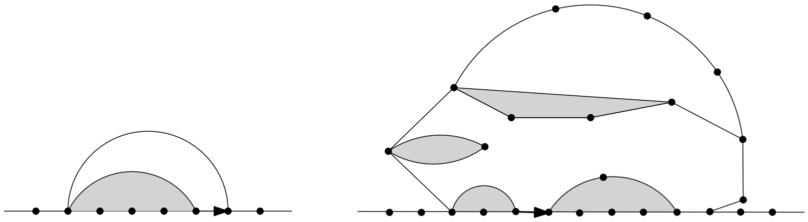
Half planar UIPT/Q satisfy translation invariance and domain Markov property.

Domain Markov Property



Q $\quad$ M $\quad$ $\tilde{M}$

Possible definitions of Domain Markov property



Distribution of the holes

- are arbitrary. No information given.
- Independent of the infinite part.
- depends on the boundary size.
- are independent.

Question: Can we characterize all half planar maps satisfying DMP and TI?

Theorem (Angel, R. 2013)

All translation invariant, domain Markov probability measures on half planar triangulations without self loops *form a one parameter family of measures with the parameter* $\alpha \in [0,1)$. α *is the probability that the triangle containing any given boundary edge is incident to an internal vertex.*

Call this measures $\mathbb{H}_\alpha$.

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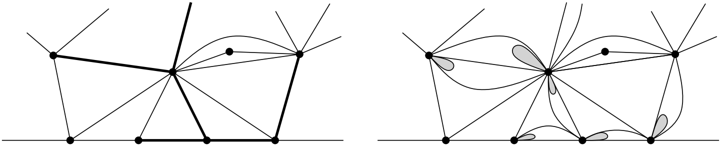

Figure : We get a family $\mathbb{H}_{\alpha,\nu,q}$ of measures.

Theorem (Angel, R. 2013)

Fix $p \geq 3$. The set of domain Markov, translation invariant probability measures on simple faced half planar p-angulations form a one parameter family. The parameter $\alpha \in [0, 1)$ and is the measure of the event that the p-gon incident to any fixed boundary edge is also incident to $p - 2$ internal vertices.

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- $\alpha = 0$: Dual of *GW* conditioned to survive.

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- $\alpha \to 1$: dual of 3-regular tree with one vertex removed.

Phase transition for triangulations.

p_i: measure of the event that the triangle incident to the root edge is incident to a vertex at distance i along the boundary.

- $\alpha < 2/3$, $p_i \approx i^{-3/2}$
- $\alpha = 2/3$, $p_i \approx i^{-5/2}$
- $\alpha > 2/3$, $p_i \approx \gamma^i$, $\gamma < 1$.

Explicit enumeration results are available!

Properties of $\mathbb{H}_\alpha$; $\alpha \in [0, 2/3)$(Subcritical)

- Tree-like structure with many cut sets of small size

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- Boundary of the ball of radius r forms a tight sequence
- Simple random walk is recurrent a.s.

Theorem (R., 2013)

Let $\alpha \in [0, 2/3)$. There exists some positive constants c, c' such that

$$\mathbb{H}_\alpha(Vol(\partial \overline{B_r}) > n) < ce^{-c'n}$$

Properties for $\mathbb{H}_\alpha$; $\alpha \in [0, 2/3)$ (Subcritical)

- Volume grows quadratically

Theorem (R., 2013)

The volume of the hull of the ball of radius r satisfies for some constants b_r and a_r

$$\frac{Vol(\overline{B_r}) - b_r}{a_r} \to Y$$

in distribution. $Y \sim Stable(1/2)$ *and* $a_r \asymp r^2$ *and* $b_r \asymp r^2$.

Properties for $\mathbb{H}_\alpha$; $\alpha \in (2/3, 1)$(Supercritical)

- Exponential Volume Growth

Theorem (R., 2013)

There exists $C > 0$ such that almost surely for r large enough,

1. $C^{-1} < \liminf_r \frac{\log Vol(\overline{B_r})}{r} < \limsup_r \frac{\log Vol(\overline{B_r})}{r} < C$
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Finite approximations

Is there a finite approximation in local weak topology similar to HUIPT for $\mathbb{H}_\alpha$?

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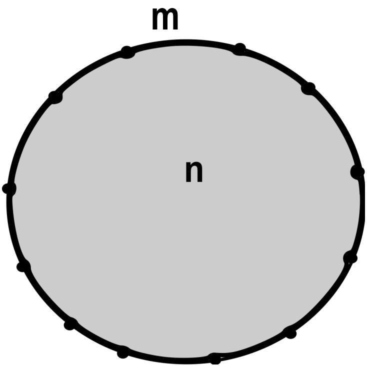
The Model:



$m/n \to a$ where $a \in [0, \infty]$

Finite approximation

Theorem (Angel,R., 2013)

Suppose $m_l/n_l \to a$ *for some number* $a \in [0, \infty]$*. Then* $\mu_{m_l,n_l} \to \mathbb{H}_\alpha$ *where* $\alpha = 2(2a+3)^{-1}$*.*

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- $n_l \gg m_l$ gives us HUIPT
- $m_l \gg n_l$ gives us $\mathbb{H}_0$ (dual of GW conditioned to survive).

Percolation (Site)

For half-plane as well as full plane UIPT, $p_c = 1/2 = p_u$ almost surely (Angel).

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Theorem (R., 2013)

For $\alpha > 2/3$,

- $p_c = \frac{1}{2}\left(1 - \sqrt{3 - \frac{2}{\alpha}}\right)$
- $p_u = \frac{1}{2}\left(1 + \sqrt{3 - \frac{2}{\alpha}}\right) > p_c$
- *Almost surely, the density of clusters on the boundary is positive.*
- *The subgraph formed by each infinite cluster has no isolated end and has continuum many ends $\mathbb{P}_p$ almost surely if $p \in (p_c, p_u)$.*

Finite approx for supercritical?

Idea: Maps of high genus with a boundary.

Summary

$\alpha \in [0, 2/3)$	Subcritical	$\lvert B_r \rvert \approx r^2$	RW recurrent
$\alpha = 2/3$	Critical:HUIPT	$\lvert B_r \rvert \approx r^4$	RW recurrent ??
$\alpha \in (2/3, 1)$	Supercritical	$\lvert B_r \rvert \approx c^r, c > 1$	RW transient