

Duality between the Ising model & 3d Quantum Gravity

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Journée Cartes - November '15



Work with F. Costantino & V. Bonzom - [arXiv:1504.02822](https://arxiv.org/abs/1504.02822) [math-ph]

& work in progress with C. Charles & J. Ben Geloun



Duality between 2d Ising and 3d Quantum Gravity

3d quantum gravity
as Ponzano-Regge spinfoam

Amplitudes as $3nj$
spin recoupling symbols

Generating function
on given graph
as Gaussian integral

Computed as
Loop Expansion
on graph

2d Ising model
partition function

Duality between
2d Ising & 3d QG

Duality between 2d Ising and 3d Quantum Gravity

Result: On a planar 3-valent graph:

- Generating function for spin network evaluations as Gaussian integral
- 2d Ising partition function as odd-Grassmann Gaussian integral
- « Equality » between the two functions, realized through supersymmetry

Applications:

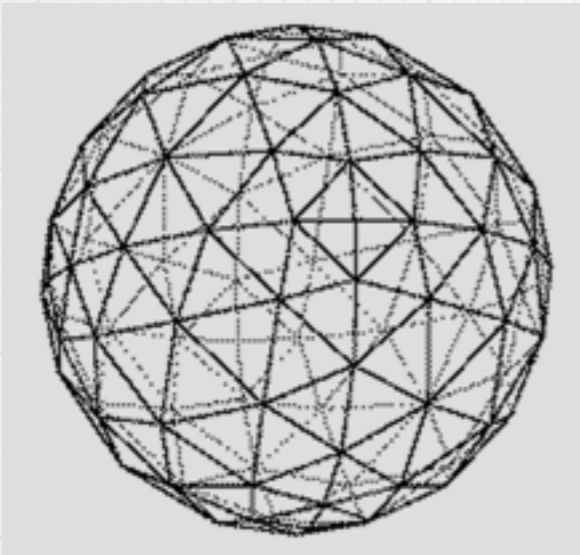
- Import statistical physics tools to QG: criticality, phase diagrams, continuum limit
- Geometrical interpretation of Ising critical couplings (Fisher zeroes)
- 3d QG & Ising as toy models for 4d gravity and more

Duality between 2d Ising and 3d Quantum Gravity

- Outline:**
1. 3d QG Ponzano-Regge amplitudes as spin network evaluations
 2. Generating function for spin networks: integral and result
 3. Ising partition function: fermionic integral & loop expansion
 4. Westbury theorem & Supersymmetry
 5. Higher order supersymmetric theories
 6. Link between Ising criticality and spin network saddle points
 7. Application to tetrahedron graph, Fisher zeroes and $6j$ duality
 8. Coarse-graining Ising, Pachner moves & Recursion relations
 9. About the continuum limit & boundary CFT for 3d QG

3d Quantum Gravity: Spinfoams & Spin Networks

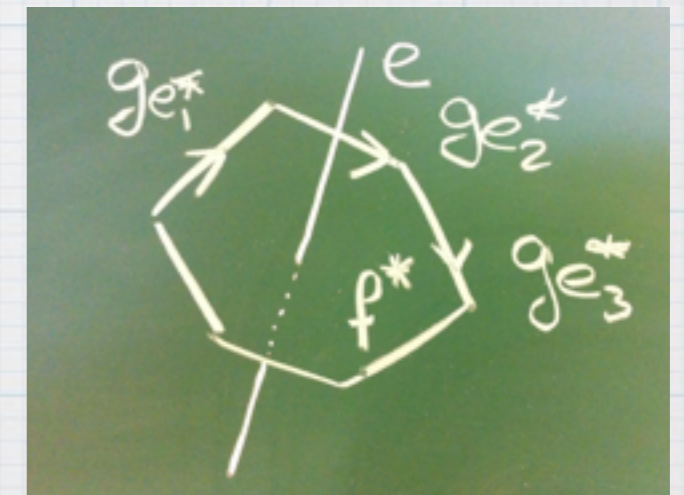
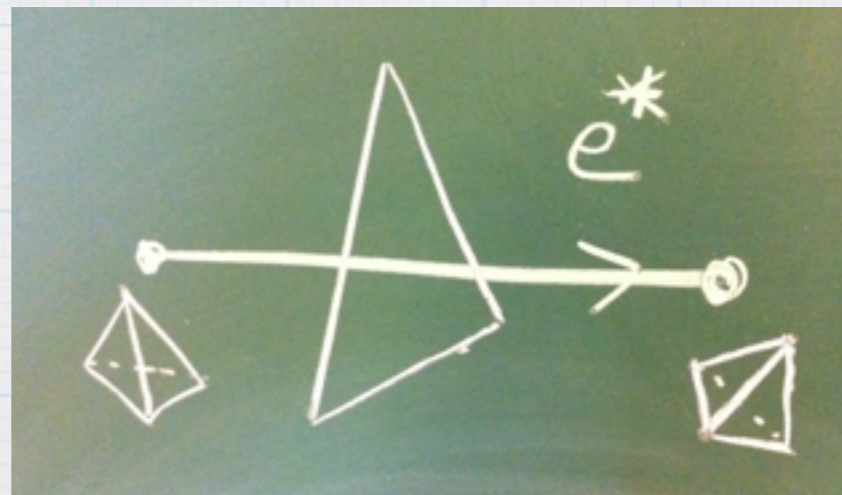
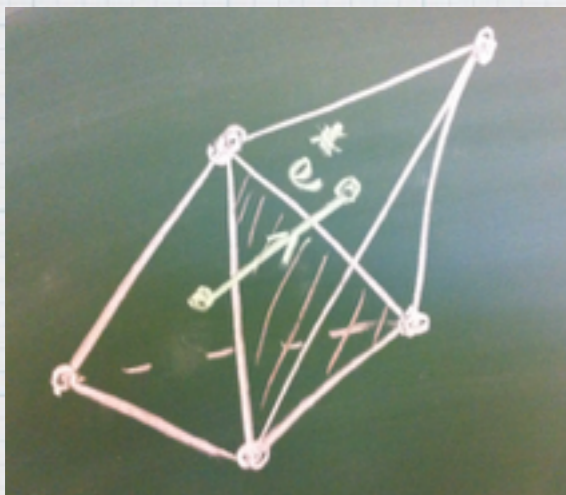
3d gravity as a TQFT can be exactly spinfoam quantized:



- 3d bulk triangulations or dual 2-complex
- Spins - Irreps of $SU(2)$ - on edges j_e
- Spin give edge length in Planck units
- Amplitude defined from $SU(2)$ recoupling

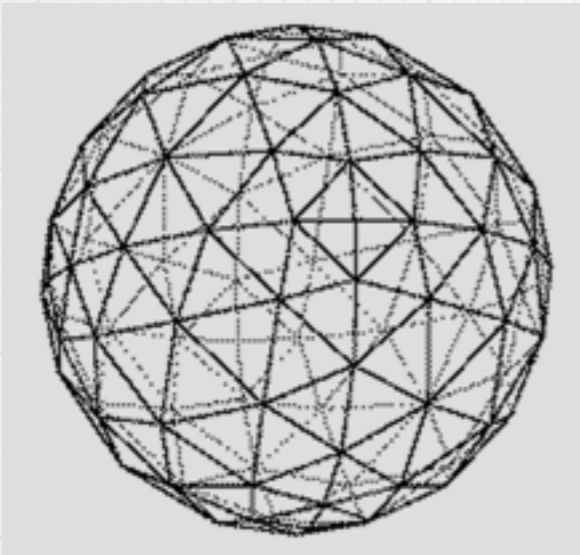
Amplitude as product of 6j-symbols

$$A_{\Delta} = \sum_{\{j_e\}} \prod_e (2j_e + 1) \prod_T \{6j\}$$



3d Quantum Gravity: Spinfoams & Spin Networks

3d gravity as a TQFT can be exactly spinfoam quantized:

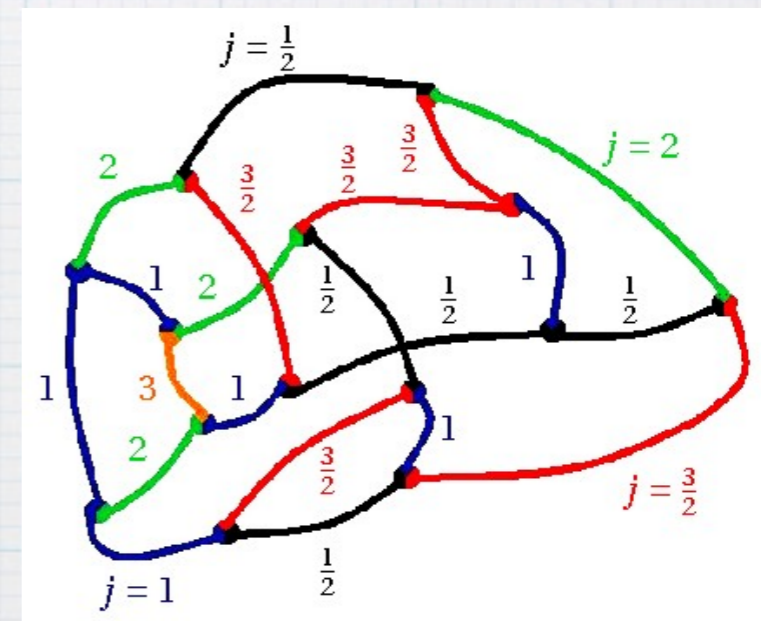


- Assume trivial spherical topology
- Use topological invariance to gauge fix bulk
- PR amplitude becomes projector on flat connection

$$\mathcal{A}_\Delta = \mathcal{A}_{\partial\Delta} = \langle \mathbb{1} | \psi \rangle = \psi(\mathbb{1})$$

For a trivial topology, amplitude expressed explicitly in terms of boundary data:

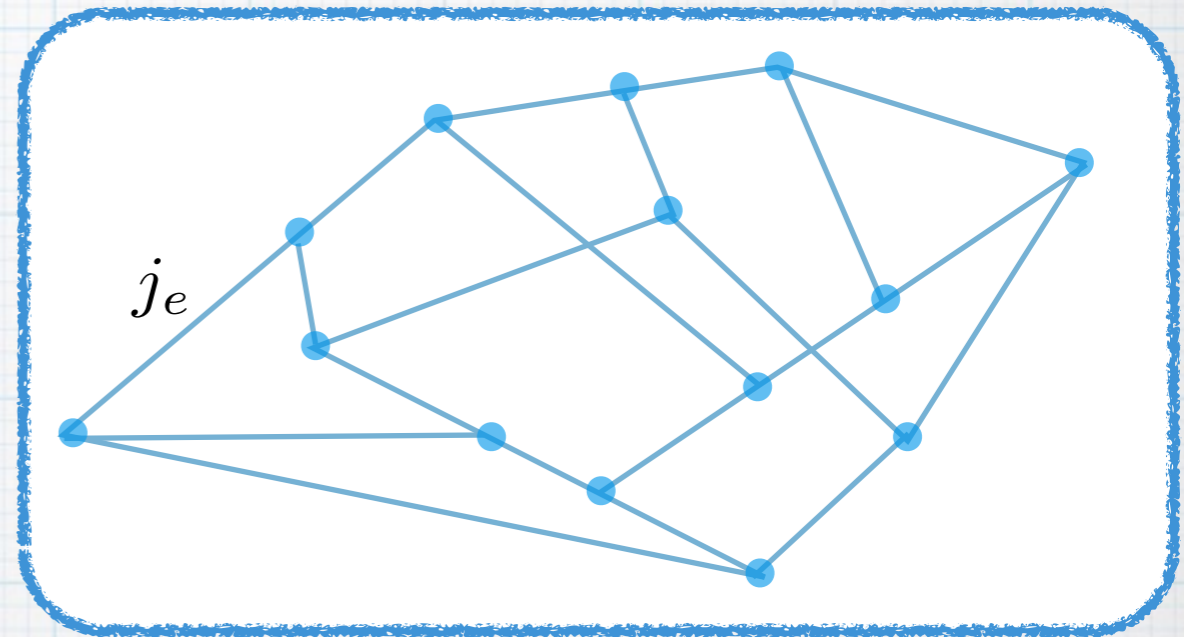
evaluation of boundary spin network



Spin Networks Evaluations

Consider **3-valent planar connected oriented boundary graph**

Spin network evaluation is a $3nj$ symbol, obtained by gluing Clebsh-Gordan coefficients:



$$s^\Gamma(\{j_e\}) = \psi_{\{j_e\}}^\Gamma(\mathbb{1}) = \sum_{\{m_e\}} \prod_e (-1)^{j_e - m_e} \prod_v \left(\begin{array}{ccc} j_{e_1}^v & j_{e_2}^v & j_{e_3}^v \\ \epsilon_{e_1}^v m_{e_1}^v & \epsilon_{e_2}^v m_{e_2}^v & \epsilon_{e_3}^v m_{e_3}^v \end{array} \right)$$

Beware of signs !

→ choose Kasteleyn orientation on planar graph to fix signs, show evaluation is independent of choice of orientation & matches standard normalizations (chromatic evaluation, unitary evaluation, ...)

Generating Function for Spin Network Evaluations

Consider **3-valent planar connected oriented boundary graph**

Define generating function for $3nj$'s using specific combinatorial weights:

$$Z_{\Gamma}^{Spin}(\{Y_e\}) = \sum_{\{j_e\}} \sqrt{\frac{\prod_v (J_v + 1)!}{\prod_{ev} (J_v - 2j_e)!}} s^{\Gamma}(\{j_e\}) \prod_e Y_e^{2j_e}$$

That's a specific choice of **boundary state with superposition of spins**

→ **Semi-classical coherent states** of geometry
with spins peaked around?

Usually, spins = length of edges of triangulation dual to graph ...
and parameters Y_e determines semi-classical geometry

Generating Function for Spin Network Evaluations

Consider **3-valent planar connected oriented boundary graph**

Define generating function for $3j$'s using specific combinatorial weights:

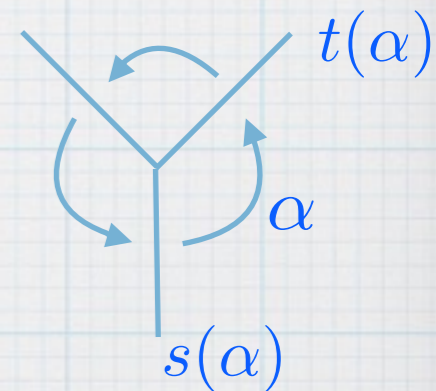
$$Z_{\Gamma}^{Spin}(\{Y_e\}) = \sum_{\{j_e\}} \sqrt{\frac{\prod_v (J_v + 1)!}{\prod_{ev} (J_v - 2j_e)!}} s^{\Gamma}(\{j_e\}) \prod_e Y_e^{2j_e}$$

Get it from gluing the $3j$ -symbol generating functions using Gaussian weights:

$$\sum_{j_e, m_e} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \sqrt{(J+1)!} \prod_e \frac{Y_e^{j_e} z_e^{j_e+m_e} w_e^{j_e-m_e}}{\sqrt{(J-2j_e)!(j_e-m_e)!(j_e+m_e)!}}$$

$$= \exp \sum_{\alpha} X_{\alpha} (z_{s(\alpha)} w_{t(\alpha)} - w_{s(\alpha)} z_{t(\alpha)})$$

$$X_{\alpha} = \sqrt{Y_{s(\alpha)} Y_{t(\alpha)}}$$



Choose cyclic orientation (anti-clockwise) around each vertex

Generating Function for Spin Network Evaluations

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Get it from gluing the $3j$ -symbol generating functions using Gaussian weights:

$$Z_{\Gamma}^{Spin}(\{Y_e\}) = \int \prod_{ev} \frac{d^2 z_{ev} d^2 w_{ev}}{\pi^2} e^{-\sum_{ev} (|z_{ev}|^2 + |w_{ev}|^2)} e^{-\sum_e (\bar{z}_{s(e)} \bar{w}_{t(e)} - \bar{w}_{s(e)} \bar{z}_{t(e)}) + \sum_{\alpha} X_{\alpha} (z_{s(\alpha)} w_{t(\alpha)} - w_{s(\alpha)} z_{t(\alpha)})}$$

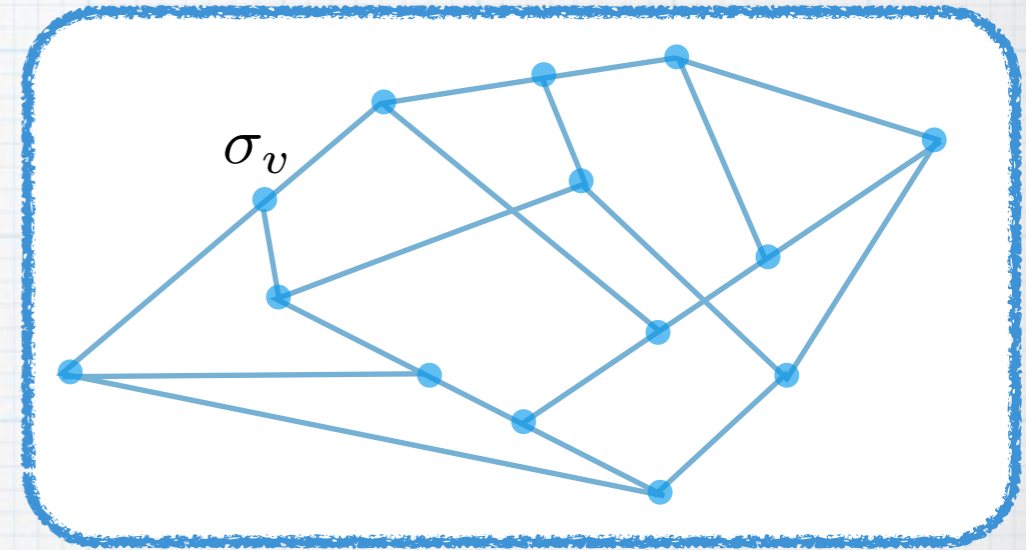
It's a Gaussian integral!!

« Simply » have to compute the determinant ...

The Ising Model Partition Function

On same graph, put « spins » on vertices: $\sigma_v = \pm 1 \in \mathbb{Z}_2$

$$Z_{\Gamma}^{Ising}(\{y_e\}) = \sum_{\sigma} \exp \left(\sum_e y_e \sigma_{s(e)} \sigma_{t(e)} \right)$$



Can define high temperature expansion...

$$Z_{\Gamma}^{Ising}(\{y_e\}) = \left(\prod_e \cosh(y_e) \right) \sum_{\sigma} \prod_e (1 + \tanh(y_e) \sigma_{s(e)} \sigma_{t(e)})$$

... as sum over loops:

$$Z_{\Gamma}^{Ising}(\{y_e\}) = 2^V \left(\prod_e \cosh(y_e) \right) \sum_{\gamma \in \mathcal{G}} \prod_{e \in \gamma} Y_e \quad \text{with } Y_e = \tanh y_e$$

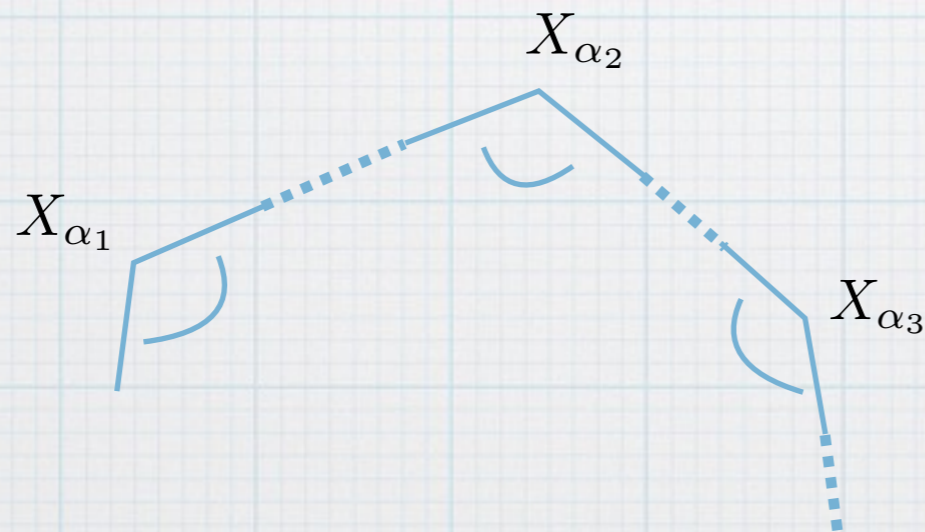
The Ising Model as a Fermion Path Integral

Two-level system naturally represented in terms of fermions.

Here explicitly: $Z_{\Gamma}^{Ising}(\{y_e\}) = 2^V \prod_e \cosh(y_e) Z_f(\{X_{\alpha}\})$

$$Z_f(\{X_{\alpha}\}) = \int \prod_{ev} d\psi_{ev} \exp \left(\sum_e \psi_{s(e)} \psi_{t(e)} + \sum_{\alpha} X_{\alpha} \psi_{s(\alpha)} \psi_{t(\alpha)} \right)$$

→ We glue angles together to form loops



Have to keep track of all the signs!

The Ising Model as a Fermion Path Integral

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→ **We glue angles together to form loops**

And for our purpose:

$$(Z_f)^2 = \int \prod_{ev} [d\psi d\eta d\bar{\psi} d\bar{\eta}]_e^v e^{\sum_{e,v} \psi_e^v \bar{\eta}_e^v + \bar{\psi}_e^v \eta_e^v} \\ e^{-\sum_e \bar{\psi}_{s(e)} \bar{\psi}_{t(e)} + \bar{\eta}_{s(e)} \bar{\eta}_{t(e)}} e^{\sum_{\alpha} X_{\alpha} (\psi_{s(\alpha)} \psi_{t(\alpha)} + \eta_{s(\alpha)} \eta_{t(\alpha)})}$$

Matching Loop Expansions

All these Gaussian integrals can be computed explicitly !

$$(Z_f)^2 Z_{\Gamma}^{Spin} = 1 \qquad Z_f = \sum_{\gamma \in \mathcal{G}} \prod_{\alpha \in \gamma} X_{\alpha} = \sum_{\gamma \in \mathcal{G}} \prod_{e \in \gamma} Y_e$$

$$(Z^{Ising})^2 Z^{Spin} = 2^{2V} \prod_e \cosh(y_e)^2$$

→ **Duality between Ising model & Spin Evaluations**

But we would like to show this without
explicitly computing those integrals !

Duality through Supersymmetry

We can introduce a meta-theory combining

- Ising model \longleftrightarrow Fermions
- Spin networks \longleftrightarrow Bosons

$$Z_{\Gamma} = (Z_f)^2 Z^{Spin} = \int dz dw d\psi d\eta e^{S[\{z,w,\psi,\eta\}_{ev}]}$$

$$S = \sum_{e,v} \lambda_{e,v} K_{e,v} + \sum_e \mu_e S_e - \sum_{\alpha} X_{\alpha} S_{\alpha}$$

$$\left| \begin{array}{l} K_{e,v} \\ S_e \\ S_{\alpha} \end{array} \right. = \begin{array}{l} |z_e^v|^2 + |w_e^v|^2 - \psi_e^v \bar{\eta}_e^v - \bar{\psi}_e^v \eta_e^v \\ \bar{z}_{s(e)} \bar{w}_{t(e)} - \bar{w}_{s(e)} \bar{z}_{t(e)} + \bar{\psi}_{s(e)} \bar{\psi}_{t(e)} + \bar{\eta}_{s(e)} \bar{\eta}_{t(e)} \\ z_{s(\alpha)} w_{t(\alpha)} - w_{s(\alpha)} z_{t(\alpha)} + \psi_{s(\alpha)} \psi_{t(\alpha)} + \eta_{s(\alpha)} \eta_{t(\alpha)} \end{array}$$

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$$S = \sum_{e,v} \lambda_{e,v} K_{e,v} + \sum_e \mu_e S_e - \sum_{\alpha} X_{\alpha} S_{\alpha}$$

We define a supersymmetry generator, acting on each half-edge $i = (ev)$:

$$\left. \begin{aligned} Qz_i &= \psi_i \\ Qw_i &= \eta_i \\ Q\psi_i &= w_i \\ Q\eta_i &= -z_i \end{aligned} \right|$$

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$$S = \sum_{e,v} \lambda_{e,v} K_{e,v} + \sum_e \mu_e S_e - \sum_\alpha X_\alpha S_\alpha$$

All terms are both Q-closed & Q-exact: $QK_{e,v} = QS_e = QS_\alpha = 0$

$$\left| \begin{array}{l} K_{e,v} = Q(\psi\bar{w} - \eta\bar{z}) \\ S_e = Q(\bar{z}\bar{\psi} + \bar{w}\eta) \\ S_\alpha = Q(z\psi + w\eta) \end{array} \right. \longrightarrow \frac{\partial \mathcal{Z}_\Gamma}{\partial \lambda_{e,v}} = \frac{\partial \mathcal{Z}_\Gamma}{\partial \mu_e} = \frac{\partial \mathcal{Z}_\Gamma}{\partial X_\alpha} = 0$$

What to do with this Ising - Spin Network duality ?

Applications:

- Map spin averages to Ising correlations
- Higher order supersymmetric actions
- Phase diagram and critical Ising couplings
- Continuum Limit of QG Amplitudes

Mapping Spin Averages to Ising correlations

Compare spin insertions in both partition functions :

$$\langle \sigma_{v_1} \sigma_{v_2} \cdots \sigma_{v_n} \rangle = \frac{1}{Z^{Ising}} \sum_{\sigma} \sigma_{v_1} \sigma_{v_2} \cdots \sigma_{v_n} e^{\sum_e y_e \sigma_s(e) \sigma_t(e)}$$

$$\langle j_{e_1}^{n_1} j_{e_2}^{n_2} \cdots j_{e_k}^{n_k} \rangle = \frac{1}{Z^{Spin}} \sum_{\{j_e\}} j_{e_1}^{n_1} j_{e_2}^{n_2} \cdots j_{e_k}^{n_k} s(\Gamma, \{j_e\}) \mathcal{W}(\{j_e\}) \prod_e (\tanh y_e)^{2j_e}$$

Can get general relation :

$$\langle j_e \rangle = \sinh y_e (\sinh y_e - \cosh y_e \langle \sigma_{s(e)} \sigma_{t(e)} \rangle)$$

$$\langle \sigma_v \sigma_w \rangle_c^{(\mathcal{P})} = \frac{-2^{n-1}}{\prod_{e \in \mathcal{P}} \sinh(2j_e)} \langle \prod_{e \in \mathcal{P}} (2j_e) \rangle_c^{(\mathcal{P})}$$

Mapping Spin Averages to Ising correlations

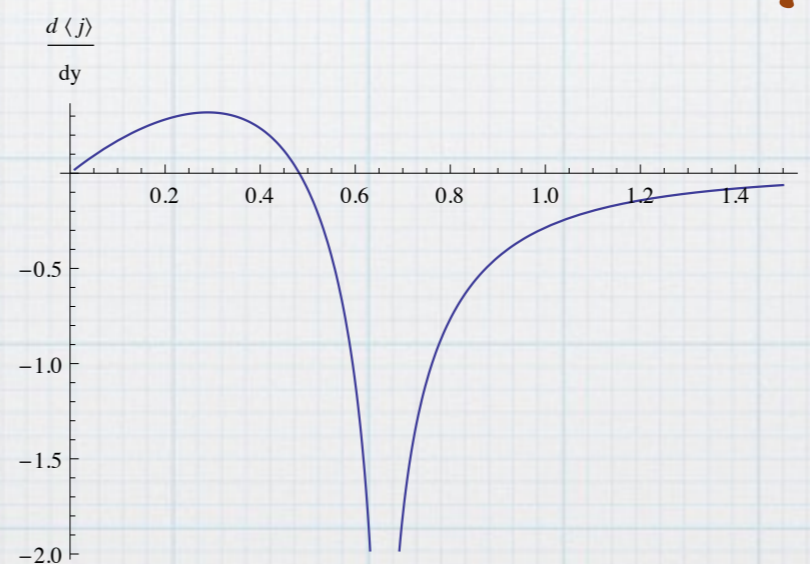
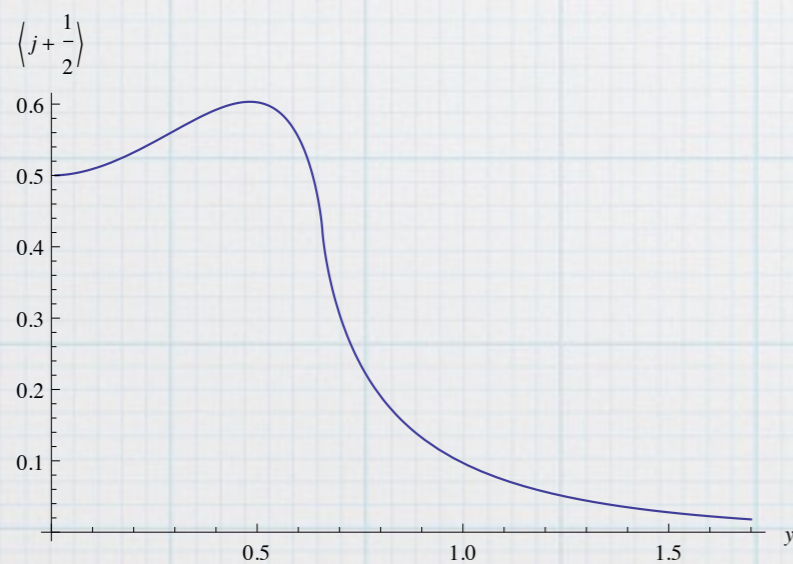
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$$\langle j_{e_1}^{n_1} j_{e_2}^{n_2} \cdots j_{e_k}^{n_k} \rangle = \frac{1}{Z^{Spin}} \sum_{\{j_e\}} j_{e_1}^{n_1} j_{e_2}^{n_2} \cdots j_{e_k}^{n_k} s(\Gamma, \{j_e\}) \mathcal{W}(\{j_e\}) \prod_e (\tanh y_e)^{2j_e}$$

Get exact formula for spin average (Baxter):

Phase Transition !!



Higher order Supersymmetric Theories and Integrals

We can go **beyond Gaussian integrals** with a quadratic action !

Up to now, we have decoupled Ising & Spin networks....

So we introduce **higher order susy interaction terms** !

→ $K_{e,v}^n, S_e^n, S_\alpha^n$ terms still supersymmetric

Adding higher order angle terms

- affects the spin network distribution & modifies saddle point (geometry background)
- geometric-dependent coupling for Ising
- couples the two Ising models

What's the physical meaning of those theories ?

Critical Ising & Spin Network Saddle Points

Let's come back to the combinatorial definition of the generating function of spin network evaluations:

$$Z_{\Gamma}^{Spin}(\{Y_e\}) = \sum_{\{j_e\}} \sqrt{\frac{\prod_v (J_v + 1)!}{\prod_{ev} (J_v - 2j_e)!}} s^{\Gamma}(\{j_e\}) \prod_e Y_e^{2j_e}$$

Spin distribution defined by statistical weight ?

$$\rho(\{j_e\}) = \sqrt{\frac{\prod_v (J_v + 1)!}{\prod_{ev} (J_v - 2j_e)!}} \prod_e Y_e^{2j_e}$$

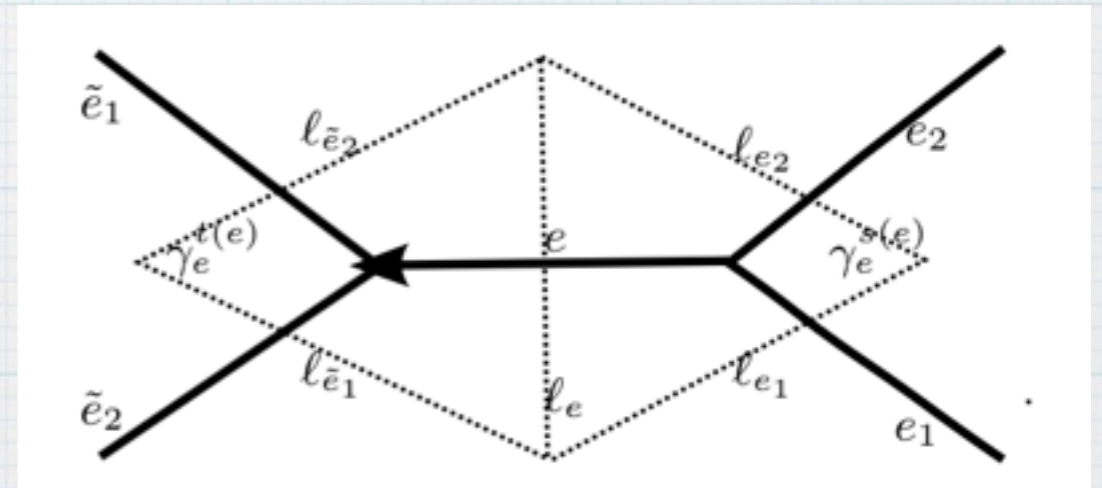
Saddle point? Geometrical Interpretation?

Critical Ising & Spin Network Saddle Points

- We proceed « as usual » :
- Large spin approx, Stirling formula
 - Look for stationary point(s)
 - Interpret spins as lengths

We get a stationary point when spins j_e are length of a triangulation if the edge couplings Y_e are determined by the condition in terms of the triangulation angles:

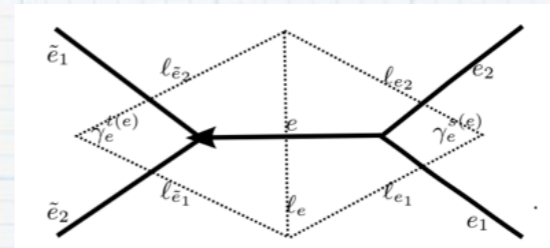
$$Y_e^2 = \tan \frac{\gamma_e^{s(e)}}{2} \tan \frac{\gamma_e^{t(e)}}{2}$$



Critical Ising & Spin Network Saddle Points

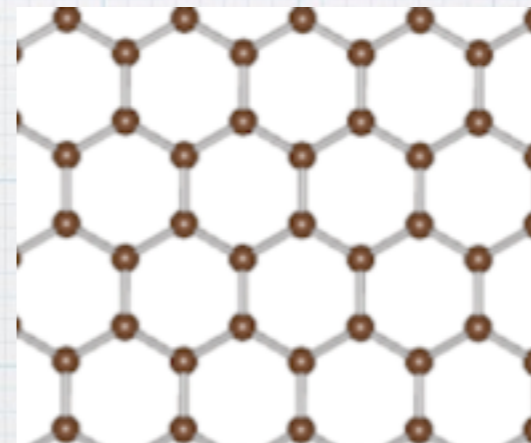
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- Regular honeycomb network

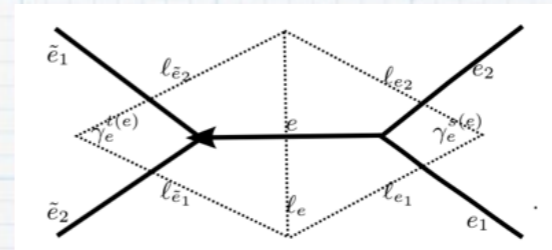
$$Y = \frac{1}{\sqrt{3}} = Y^{critical}$$



Critical Ising & Spin Network Saddle Points

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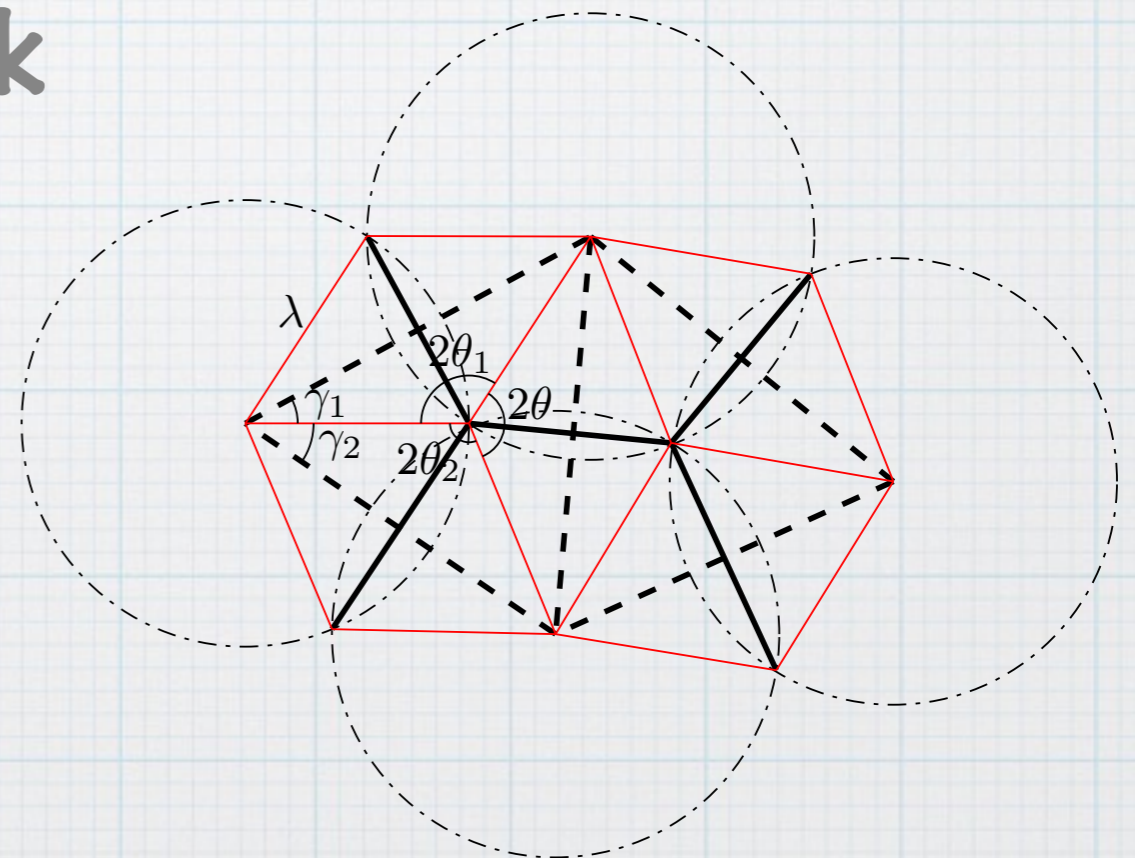


- Regular honeycomb network

$$Y = \frac{1}{\sqrt{3}} = Y^{critical}$$

- Also isoradial graphs !

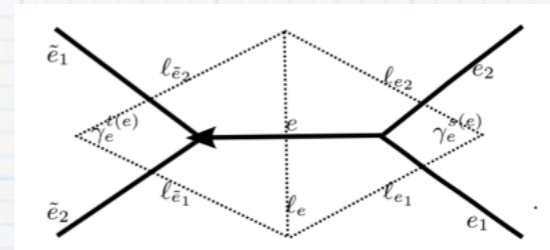
$$Y_e^c = \tan \frac{\gamma_e}{2} = \tan \frac{\theta_e}{2}$$



Critical Ising & Spin Network Saddle Points

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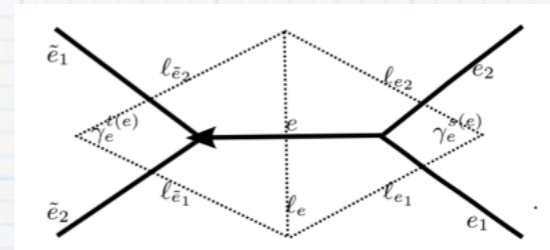
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More general !?!

Critical Ising & Spin Network Saddle Points

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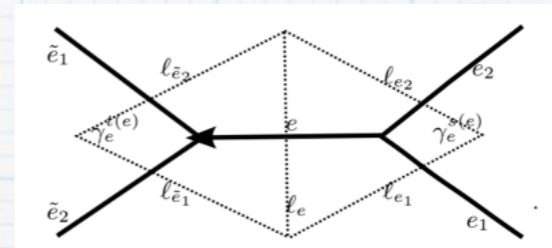


Admissible geometric
couplings Y_e^{geom}

Critical Ising & Spin Network Saddle Points

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Admissible geometric couplings Y_e^{geom}

Scale invariant saddle points in spins j_e

Pole in the generating function $Z^{Spin} \rightarrow \infty$

Zero of Ising partition function i.e. critical couplings Y_e^c

The Tetrahedron & the 6j Symbol

Test all this on the Tetrahedron !

- Look at generating function for 6j symbols
- Study saddle points of combining both weight & 6j symbol with Regge action at large spins
- Provide geometrical interpretation for Fisher zeroes on tetrahedron graph

$$\left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{matrix} \right\} \underset{j \rightarrow \infty}{\sim} \frac{1}{\sqrt{12\pi V}} \cos\left(\sum_{k=1}^6 (j_k + \frac{1}{2})\theta_k + \frac{\pi}{4}\right)$$

The Tetrahedron & the $6j$ Symbol

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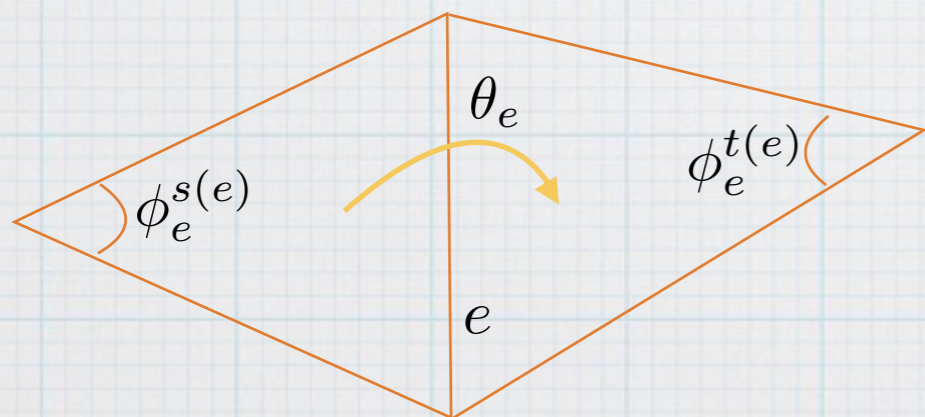
Work with
V. Bonzom & C. Charles
to appear soon

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Critical couplings for Ising are complex, with phase given by dihedral angles



$\epsilon = \pm$ global sign

$$Y_e^c = e^{\epsilon \frac{i}{2} \theta_e} \sqrt{\tan \frac{\phi_e^{s(e)}}{2} \tan \frac{\phi_e^{t(e)}}{2}}$$

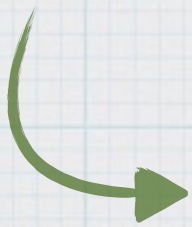
only depends on geometry
up to global scale factor !

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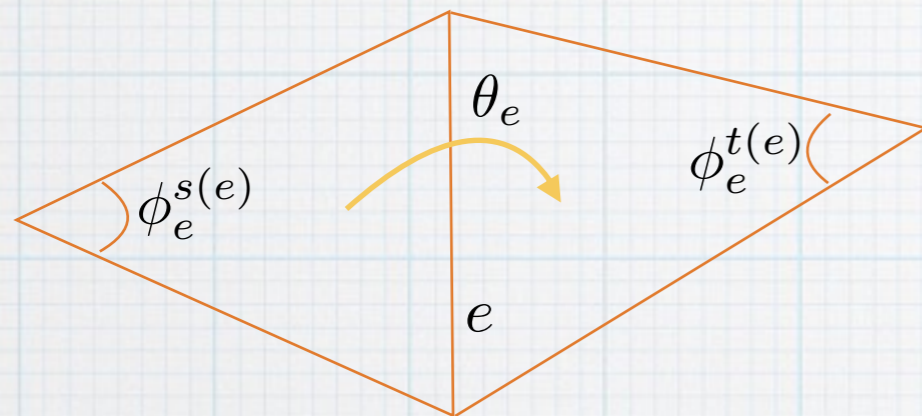


Phase represents extrinsic curvature of surface in 3d space

$$Y_e^c = e^{\epsilon \frac{i}{2} \theta_e} \sqrt{\tan \frac{\phi_e^{s(e)}}{2} \tan \frac{\phi_e^{t(e)}}{2}}$$

The Tetrahedron & the 6j Symbol

Critical couplings for Ising are complex, with phase given by 3d dihedral angles and modulus given by 2d triangle angles



$$Y_e^c = e^{\epsilon \frac{i}{2} \theta_e} \sqrt{\tan \frac{\phi_e^{s(e)}}{2} \tan \frac{\phi_e^{t(e)}}{2}}$$

These are roots of the tetrahedron loop polynomial :

$$P[Y_e] = 1 + Y_1 Y_2 Y_6 + Y_1 Y_3 Y_5 + Y_2 Y_3 Y_4 + Y_4 Y_5 Y_6 + Y_1 Y_4 Y_2 Y_5 + Y_2 Y_5 Y_3 Y_6 + Y_1 Y_4 Y_3 Y_6$$

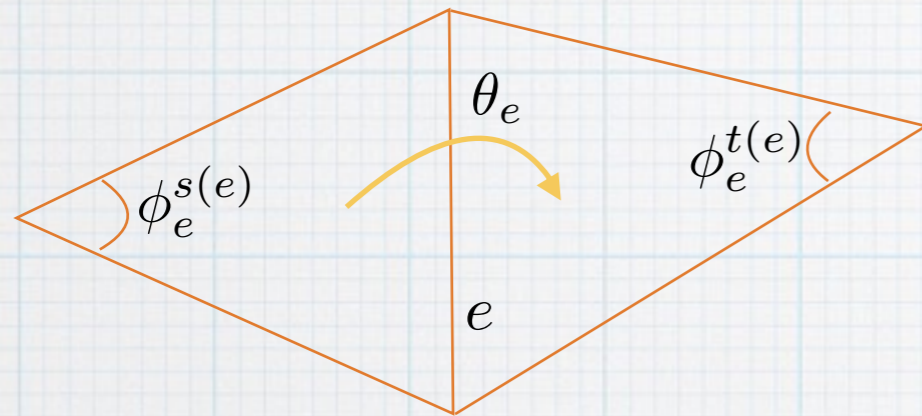
Direct proof from spherical trigonometry

but this only gives a 5d manifold

within the 10d space of solutions

The Tetrahedron & the 6j Symbol

Critical couplings for Ising are complex, with phase given by 3d dihedral angles and modulus given by 2d triangle angles



$$Y_e^c = e^{\epsilon \frac{i}{2} \theta_e} \sqrt{\tan \frac{\phi_e^{s(e)}}{2} \tan \frac{\phi_e^{t(e)}}{2}}$$

These are roots of the tetrahedron loop polynomial :

$$P[Y_e] = 1 + Y_1 Y_2 Y_6 + Y_1 Y_3 Y_5 + Y_2 Y_3 Y_4 + Y_4 Y_5 Y_6 + Y_1 Y_4 Y_2 Y_5 + Y_2 Y_5 Y_3 Y_6 + Y_1 Y_4 Y_3 Y_6$$

Direct proof is painful...

and only gives a 5d manifold within the 10d space of solutions

Have to go to complex tetrahedra ! Work in progress

The Tetrahedron & the 6j Symbol

Can go deeper on tetrahedron with high \mathcal{T} /low \mathcal{T} duality

Use loop expansion of 2d Ising to show duality identity on the partition function :

High \mathcal{T} loop expansion:

$$Z_{\Gamma}(y_e) = \sum_{\{\sigma_v = \pm 1\}} e^{\sum_e y_e \sigma_{s(e)} \sigma_{t(e)}} = 2^V \prod_e \cosh y_e \sum_{C \subset \Gamma} \prod_{e \in C} \tanh y_e$$

Low \mathcal{T} cluster expansion:

$$Z_{\Gamma}(y_e) = 2 \prod_e e^{y_e} \sum_{C^* \subset \Gamma^*} \prod_{e \in C^*} e^{-2y_e}$$

The Tetrahedron & the 6j Symbol

Can have more fun on tetrahedron with high T /low T duality

Use loop expansion of 2d Ising to show duality identity on the partition function :

$$Z_{\Gamma}(y_e) = \frac{2 \prod_e e^{y_e}}{2^{V^*} \prod_e \cosh \tilde{y}_e} Z_{\Gamma^*}(\tilde{y}_e)$$

with dual couplings $Y_e = \tanh y_e = e^{-2\tilde{y}_e}$, $\tilde{Y}_e = \tanh \tilde{y}_e = e^{-2y_e}$

$$Y = \mathcal{D}(\tilde{Y}) = \frac{(1 - \tilde{Y})}{(1 + \tilde{Y})}$$

$$\tilde{Y} = \mathcal{D}(Y) = \frac{(1 - Y)}{(1 + Y)}$$

Duality transform is involution,
relating the graph and its dual

The Tetrahedron & the 6j Symbol

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Duality transform is involution,
relating the graph and its dual

$$\tilde{Y} = \mathcal{D}(Y) = \frac{(1 - Y)}{(1 + Y)}$$

Its fixed point is critical Ising
coupling for square lattice :

$$Y_c = -(1 \pm \sqrt{2})$$

The Tetrahedron & the 6j Symbol

Can have more fun on tetrahedron with high T /low T duality

Apply to 6j generating function :

$$4^3 \sum_{\{j_e\}} \left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{matrix} \right\} \prod_v \Delta_v(j_e) \prod_e (-1)^{2k_e} T(2j_e + 1, 2k_e + 1) = \left\{ \begin{matrix} k_4 & k_5 & k_6 \\ k_1 & k_2 & k_3 \end{matrix} \right\} \prod_{v^*} \Delta_{v^*}(k_e)$$

with transform coefficients given by power series (figurate numbers) :

$$Y \frac{(1 - Y)^{2j}}{(1 + Y)^{2(j+1)}} = \sum_{k \in \mathbb{N}/2} (-1)^{2k} T(2j + 1, 2k + 1) Y^{2k+1}$$

Could be related to self-duality of squared q -deformed 6j symbol ...

The Tetrahedron & the 6j Symbol

Lessons from the tetrahedron :

- Geometric characterization of Fisher zeroes for the Ising model : graph is planar but not flat, critical couplings defined by 3d embedding (dihedral angles)
- Low T / High T Ising duality gives relation between graph and dual graph for spin networks : non-perturbative relations for spinfoams? another path towards criticality ?

From Coarse-Graining Ising to boundary Pachner moves

Natural application of duality between Ising models & spin networks :

COARSE-GRAINING

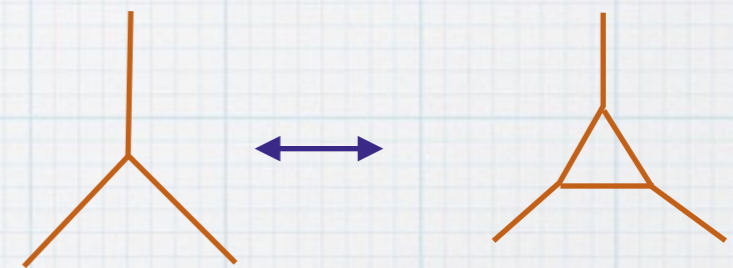
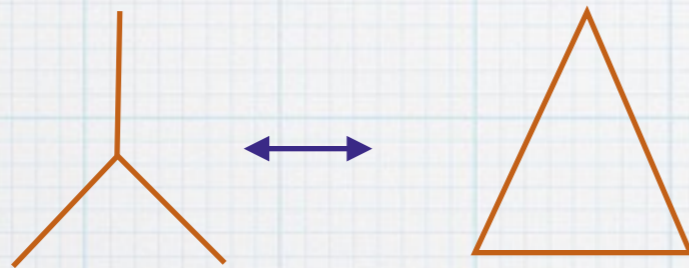
Work with
J. Ben Geloun
to appear soon

Star-Triangle relation

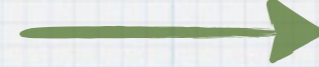


3-1 Pachner move

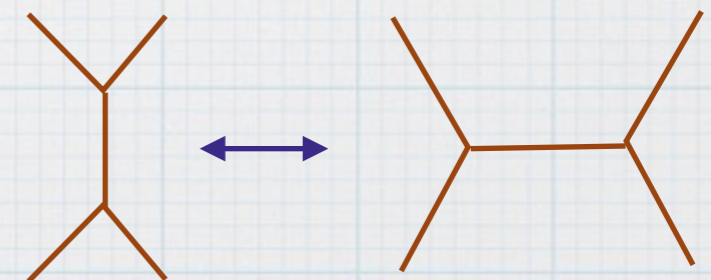
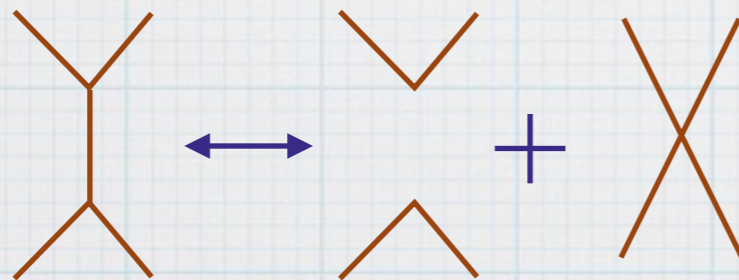
Related to
Yang-Baxter
equation



(multivariate) Tutte recursion relation



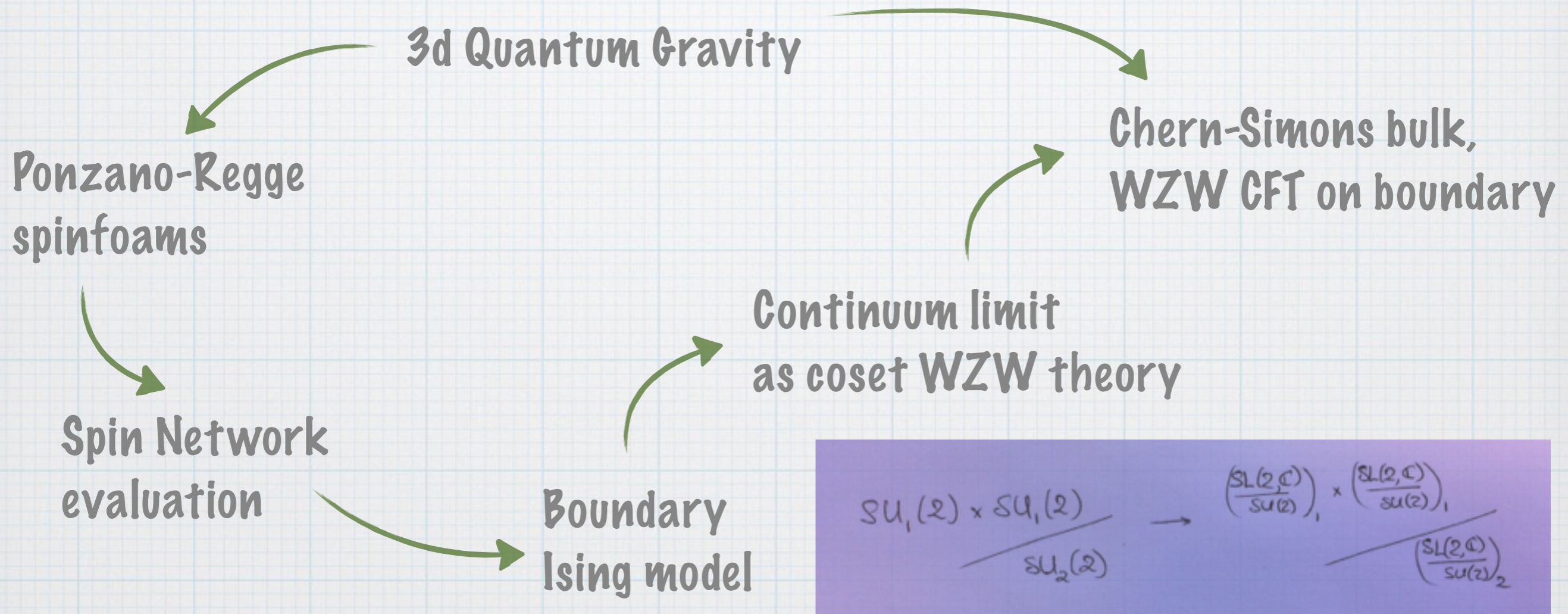
2-2 Pachner move



Continuum limit and boundary CFT

Use known continuum limit of Ising models to derive boundary CFT description of Ponzano-Regge spinfoam models at critical point

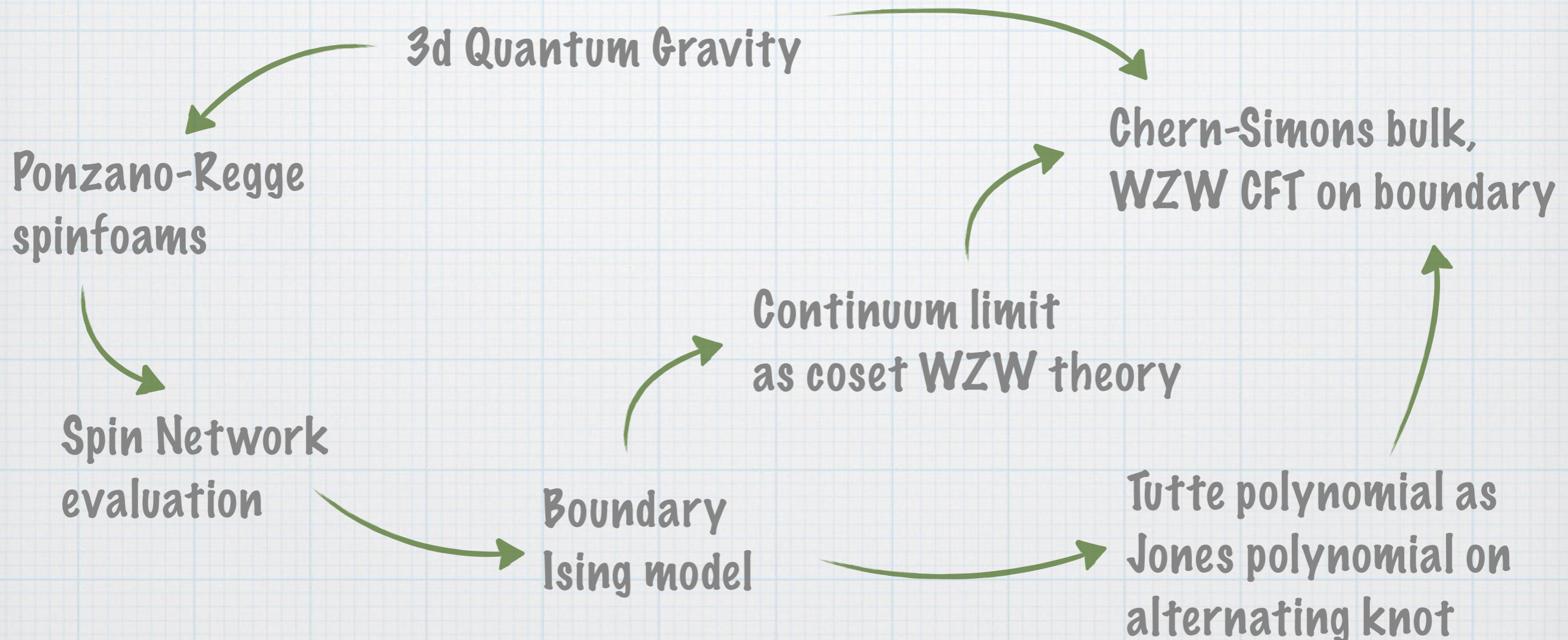
Let's try to close the loop :



Continuum limit and boundary CFT

Use known continuum limit of Ising models to derive boundary CFT description of Ponzano-Regge spinfoam models at critical point

Let's try to close the loop :



Ising-QG Duality: Extensions & Prospects

- Improvements: arbitrary valence, non-planar graphs, holonomy insertions, q -deformation, dual Potts model, mag field (L-Y theorem)?
- Full saddle points for arbitrary graphs towards geometric characterization of Fisher zeroes
- Meaning of higher order susy models and localized integrals, Compare with supergravity (Casson invariant & $osp(2|1)$ spinnets)
- Apply to Spin glasses ?
- Linked to the relation between squared critical Ising & dimers?
- Sum over planar triangulations à la Kazakov (matrix models & GFT)
- Continuum limit of Ising model as WZW coset model, boundary CFT for spinfoams & models for conformal gravity
- Ising duality for 3d gravity on AdS (CFT and BMS symmetry)

Duality between 2d Ising and 3d Quantum Gravity

Thank you for your attention !!

3d Quantum Gravity: Spinfoams & Spin Networks

3d gravity as a TQFT:

$$S[A, e] = \int_{\mathcal{M}} \text{Tr } e \wedge F[A] = \int_{\mathcal{M}} \delta_{ij} \epsilon^{abc} e_a^i F_{bc}^j[A]$$

- Triad e 1-form with value in $\mathfrak{su}(2)$ Lie algebra
- $SU(2)$ Connection A with curvature $F[A] = dA + A \wedge A$

Topological field theory with no local degrees of freedom

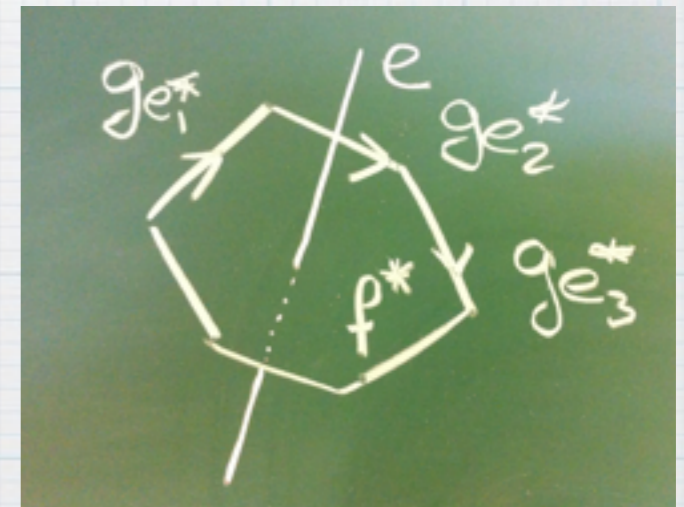
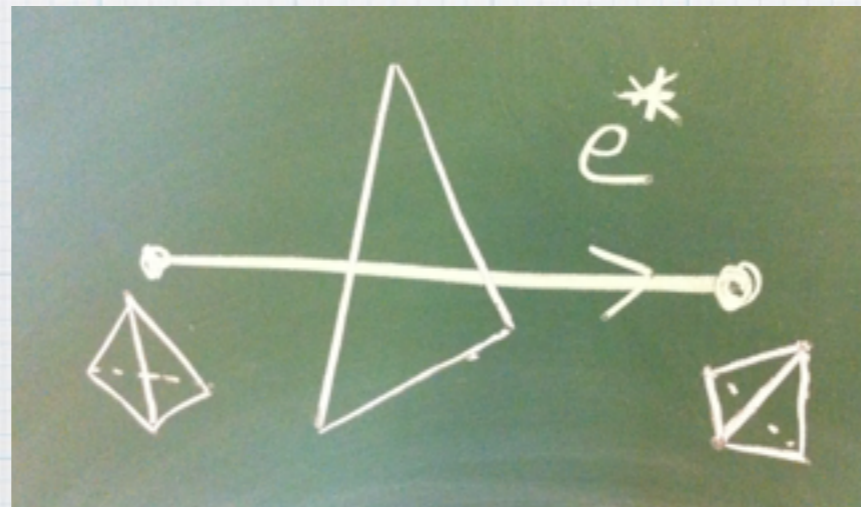
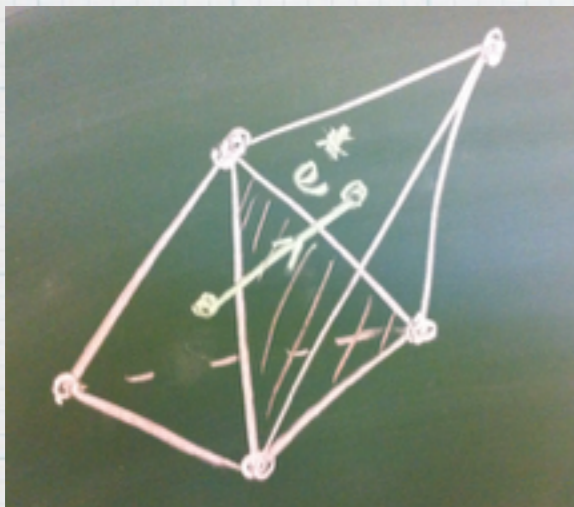
- $SU(2)$ Gauge invariant & Diffeomorphism invariant
- Theory of a pure flat connection $F[A]=0$
- If add volume term, equivalent to Chern-Simons theory

3d Quantum Gravity: Spinfoams & Spin Networks

3d gravity as a TQFT can be exactly spinfoam quantized:

Topological field theory \longrightarrow Can be discretized exactly

1. Choose a 3d triangulation (cellular decomposition works too)
2. Define dual 2-complex, the **spinfoam**
3. Discretize connection along dual edges $g_{e^*} \in \text{SU}(2)$
4. Discretize triad along edges $X_e \in \text{su}(2)$



3d Quantum Gravity: Spinfoams & Spin Networks

3d gravity as a TQFT can be exactly spinfoam quantized:

Topological field theory \longrightarrow Can be discretized exactly

- Connection along dual edges $g_{e^*} \in \text{SU}(2)$
- Triad along edges $X_e \in \mathfrak{su}(2)$

X 's are Lagrange multipliers
imposing flatness of connection
around dual faces (i.e around edges)

$$G_e = G_{f^*} = \overrightarrow{\prod_{e^* \in \partial f^*}} g_{e^*}$$

$$Z = \int \text{ded}A e^{iS[e,A]} = \int \text{d}A \delta(F[A]) = \int \prod_{e^*} \text{d}g_{e^*} \prod_e \delta(G_e)$$

3d Quantum Gravity: Spinfoams & Spin Networks

3d gravity as a TQFT can be exactly spinfoam quantized:

Topological field theory \rightarrow Can be discretized exactly

$$Z = \int \text{ded}A e^{iS[e,A]} = \int \text{d}A \delta(F[A]) = \int \prod_{e^*} \text{d}g_{e^*} \prod_e \delta(G_e)$$

We decompose onto irreps of $SU(2)$ i.e **spins** :

$$Z = \int \prod_{e^*} \text{d}g_{e^*} \sum_{\{j_e \in \frac{\mathbb{N}}{2}\}} \prod_e (2j_e + 1) \chi_{j_e}(G_e)$$

and we integrate over all group elements,
leaving us with spin recoupling symbols